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TESTING WHETHER ONE REGRESSION FUNCTION IS LARGER THAN ANOTHER

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ABSTRACT

The problem of testing whether one regression function is larger than another on a specified set R is considered. The regression functions must be linear functions of the parameters but need not be linear functions of the independent variables. The proposed test has an exactly specified size in typical situations. The test's critical value is a standard t percentile. The power function of the test is investigated.

1. INTRODUCTION

In this paper we consider testing whether the regression function from one population is above the regression function from another population for all values of the independent variable in a specified set.

Tsutakawa and Hewett (1978), Hewett and Lababidi (1980) and Spurrier, Hewett and Lababidi (1982) have previously considered

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this testing problem. Their tests assume that the regression functions are linear functions of the independent variable. Their tests cannot be used if some form other than linear is assumed for the regression functions. The model considered herein generalizes the models of Tsutakawa and Hewett (1978) and Hewett and Lababidi (1980) in three ways. First the regression functions may be functions other than linear functions of the independent variables. For example, the regression functions may be quadratic or higher degree polynomials. Second, the two regression functions need not be of the same functional form. For example, one may be a linear function and the other a quadratic function. Third, the set of values of the independent variable of interest to the experimenter need not be a hypercube as required by these two models. The test proposed herein reduces to the tests of Tsutakawa and Hewett (1978) and Hewett and Lababidi (1980) for the special models they consider.

In Section 3 an example is considered. It concerns comparison of particulate emissions (dependent variable) over a range of flue openings (independent variable) for large and small flues (populations). The models and tests of the previous authors are inappropriate for this data because a linear regression provides a poor fit for the data. But the model and test proposed in Section 2 may be used because a quadratic regression provides a good fit. Computational aspects of the test are discussed in Section 4. The results of a simulation study investigating the power of the proposed test are discussed in Section 5. Theoretical results regarding the size of the test and equivalence with other tests are proved in Section 6.

2. MODEL AND TEST

Let $\{(X_{1j}, Y_{1j}), j = 1,\ldots,n_1\}$ and $\{(X_{2j}, Y_{2j}), j = 1,\ldots,n_2\}$ denote two independent sets of observations where $X_{ij} = (X_{ij1},\ldots,X_{ijk})$. The independent variables X_{ij} may be observed random vectors or design variables fixed by the experimenter.

The entire analysis is conditioned on the observed values of X_{ij}. Let R denote a set of values of X_{ij} of interest to the experimenter. We assume that given the $X_{ij} = x_{ij}$, the dependent variables Y_{ij} are independent normal random variables with mean and variance given by

$$E(Y_{ij}|X_{ij} = x_{ij}) = \sum_{m=1}^{p_i} \beta_{im} f_{im}(x_{ij}) = f_i(x_{ij})\beta_i$$

and

$$\mathrm{Var}(Y_{ij}) = \sigma^2 .$$

$\beta_i = (\beta_{i1},\ldots,\beta_{ip_i})'$, i = 1,2, and σ^2 are unknown parameters. The $f_i(x) = (f_{i1}(x),\ldots,f_{ip_i}(x))$, i = 1,2, are known vectors of functions which define the functional form of the regression functions.

By allowing $p_1 \neq p_2$ and $f_{1m}(x) \neq f_{2m}(x)$, this model allows the two regression functions to have different functional forms. The first might be a linear function and the second a quadratic function. As mentioned in Section 1, previous models for this problem required both regression functions to be linear functions of the independent variable.

We wish to compare the regression functions $f_1(x)\beta_1$ and $f_2(x)\beta_2$. In particular we are interested in whether $f_1(x)\beta_1$ is greater than $f_2(x)\beta_2$ on R. The test we will propose is a size α test of

$$H_0:\ f_1(x)\beta_1 \le f_2(x)\beta_2 \quad \text{for at least one } x \in R$$

versus

$$H_A:\ f_1(x)\beta_1 > f_2(x)\beta_2 \quad \text{for every } x \in R.$$

Let b_1 and b_2 denote the least squares estimates of β_1 and β_2 and let

$$s^2 = \sum_{i=1}^{2} \sum_{j=1}^{n_i} (y_{ij} - f_i(x_{ij})b_i)^2/\nu$$

denote the pooled estimate of σ^2 where $\nu = n_1 - p_1 + n_2 - p_2$. The estimate b_i has a multivariate normal distribution with mean β_i and covariance matrix $\sigma^2 D_i^{-1}$ where the (m, n) element of D_i is

$$\sum_{j=1}^{n_i} f_{im}(x_{ij}) f_{in}(x_{ij}) .$$

Let $e(x) = f_1(x) D_1^{-1} f_1'(x) + f_2(x) D_2^{-1} f_2'(x)$. Then the variance of $f_1(x) b_1 - f_2(x) b_2$ is $\sigma^2 e(x)$.

The test we propose for testing H_0 versus H_A is based on the test statistic T defined by

$$T = \min_{x \varepsilon R} T(x) \tag{2.1}$$

where

$$T(x) = \frac{f_1(x) b_1 - f_2(x) b_2}{s\sqrt{e(x)}} . \tag{2.2}$$

The test rejects H_0 in favor of H_A if and only if $T > t_{1-\alpha,\nu}$ where $t_{1-\alpha,\nu}$ is the $1 - \alpha$ percentile of a t distribution with ν degrees of freedom.

This test is always a level α test in that the probability of a type one error is always less than or equal to α. This test has size exactly equal to α if (2.3), (2.4), and (2.5) are true.

R is a closed and bounded, i.e., compact, subset of k-dimensional Euclidean space. (2.3)

The functions $f_{ij}(x)$ are continuous functions on R. (2.4)

There are values of β_1 and β_2 such that $f_1(x)\beta_1 = f_2(x)\beta_2$ for one value of $x \varepsilon R$ and $f_1(x)\beta_1 > f_2(x)\beta_2$ for all other $x \varepsilon R$. (2.5)

These facts are proved in Section 6. Two examples of when (2.3), (2.4), and (2.5) are satisfied and the size is exactly α are also given in Section 6.

The test we have proposed may be motivated in this way. It rejects H_0 if and only if for each $x \varepsilon R$, the individual test of H_{0x}: $f_1(x)\beta_1 \leq f_2(x)\beta_2$ versus H_{Ax}: $f_1(x)\beta_1 > f_2(x)\beta_2$ based on the test statistic $T(x)$ rejects H_{0x}. Such tests are called intersection-union tests. They have been discussed by Gleser (1973), Berger (1982) and Berger and Sinclair (1984).

3. APPLICATION

As an application of the test proposed in Section 2, consider the air pollution data for wood stoves found in Table I. The independent variable X is the air intake setting; values of .25, .50, .75 and 1.00 = fully open were used in the study. Table I contains data from two populations corresponding to large (population 1) and small (population 2) flue sizes.

Suppose the experimenter wishes to compare the average particulate matter vented over the range of air intake settings $R = \{x: .50 \le x \le 1.00\}$. Perhaps this range is of interest because the settings people use most often are in this range. In particular suppose the experimenter wishes to determine if the average emissions from the small flue are lower than the average emissions from the large flue. If this were true then the small flue could be considered better than the large flue in controlling emissions.

The data from Table I are plotted in Figure 1. The plot indicates that a linear regression will provide a poor fit to both the large and small flue data. Indeed, $R^2 = .16$ for the large

TABLE I

Particle Emissions at Various Flue Settings

Setting (fraction open)	Emissions (percent light blocked)	
	Large flue	Small flue
.25	42, 40, 39	44, 42, 42
.50	26, 26, 30	28, 28, 27
.75	29, 27, 29	26, 27, 25
1.00	34, 33, 34	31, 36, 34

Data from Mendenhall, W. and Rienmuth J. E. (1982). Statistics for Management and Economics, Fourth Edition. Duxbury Press, Boston, 537.

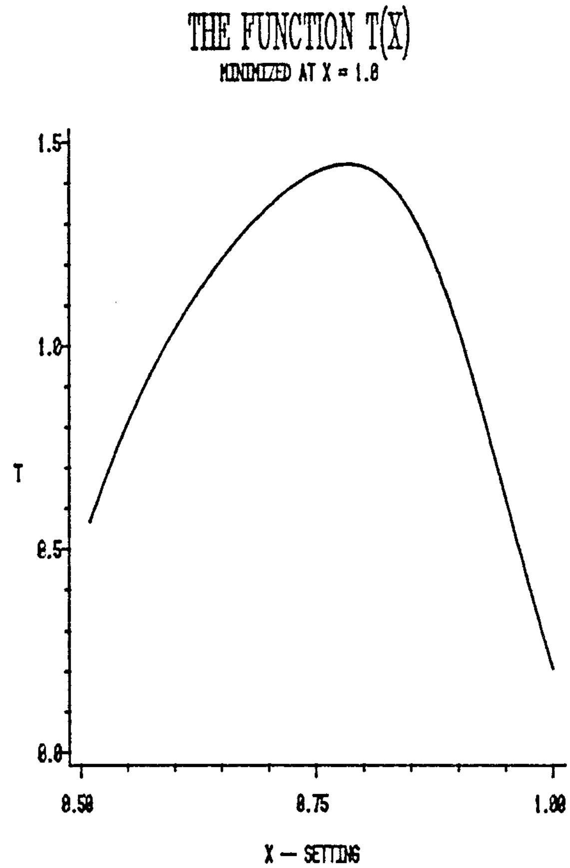


FIGURE 1

flue and R^2 = .23 for the small flue when simple linear regressions are performed. Since linear regressions provide such a poor fit, the tests of Tsutakawa and Hewett (1978), Hewett and Lababidi (1980), and Spurrier, Hewett and Lababidi (1982) are inappropriate for this problem. But quadratic regressions fit the data from both populations well. For quadratic regressions, R^2 = .90 for the large flue and R^2 = .96 for the small flue.

In an effort to determine if the small flue is better than the large flue for $.50 \leq x \leq 1.00$, consider testing

$$H_0: \ \beta_{11} + \beta_{12}x + \beta_{13}x^2 \leq \beta_{21} + \beta_{22}x + \beta_{23}x^2$$
$$\text{for some } .50 \leq x \leq 1.00$$

versus

$$H_A: \ \beta_{11} + \beta_{12}x + \beta_{13}x^2 > \beta_{21} + \beta_{22}x + \beta_{23}x^2$$
$$\text{for all } .50 \leq x \leq 1.00 \ .$$

The estimated regression functions are $60.08 - 99.27x + 73.33x^2$ for the large flue and $68.00 - 124.80x + 90.67x^2$ for the small flue. These regression functions are shown in Figure 1.
The pooled estimate of σ^2 is $s^2 = 53.08/18 = 2.95$. The correlation matrices are

$$D_1^{-1} = D_2^{-1} = \begin{pmatrix} 2.58 & -9.00 & 6.67 \\ -9.00 & 34.40 & -26.67 \\ 6.67 & -26.67 & 21.33 \end{pmatrix} .$$

Thus the test statistic T for testing H_0 versus H_A is

$$T = \min_{.5 \leq x \leq 1} T(x)$$
$$= \min_{.5 \leq x \leq 1} \frac{-7.92+25.53x-17.34x^2}{1.72\sqrt{5.16-36.00x+95.48x^2-106.68x^3+42.66x^4}} \ .$$

The function T(x) is graphed in Figure 2 and is clearly minimized for $.5 \leq x \leq 1$ at $x = 1$. Thus $T = T(1) = .21$. Using $\alpha = .10$, we find $t_{.90,18} = 1.333$. The test does not reject H_0 since $T < 1.333$. We cannot conclude that the average emissions for large flues is greater than the average emissions for small flues for all settings between .5 and 1.

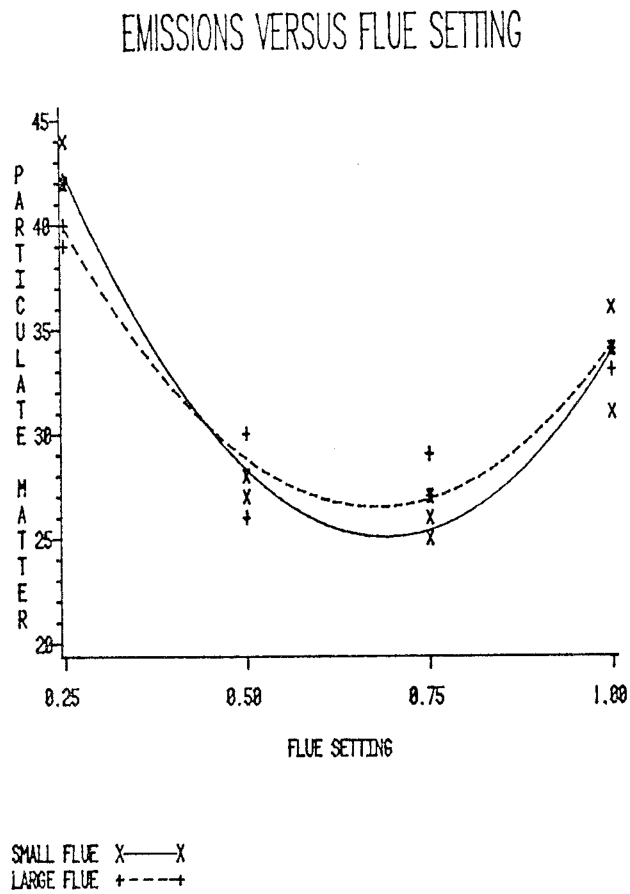


FIGURE 2

4. COMPUTATIONAL CONSIDERATIONS

In (2.1) the test statistic T was defined as the minimum of T(x) over the set R. Typically the computation of the test statistic will be accomplished by a numerical minimization of T(x). If x is univariate the evaluation of this minimum might be accomplished by means of a graph of T(x), such as Figure 2. The problem of minimizing a function such at T(x) which is the ratio of two functions of x has been studied extensively in the mathematical programming literature by Charnes and Cooper (1962), Swarup (1965), Sharma (1967) and Craven and Mond (1973, 1975a, and 1975b). These authors have found that this nonlinear programming problem is equivalent to other nonlinear programming problems which do not involve fractions. These results could simplify the numerical minimization of $T(x)$.

But to perform the test the actual value of T need not be computed. One only needs to know whether $T > t_{1-\alpha,\nu}$ or $T \leq t_{1-\alpha,\nu}$. In this section we describe two shortcuts for making this determination without the exact computation of T.

4.1 Shortcut for determining if H_0 is accepted

Let X^* denote an arbitrary finite subset of R. For example, if $R = \{x: x_{i*} \leq x_i \leq x_i^*, i = 1,\ldots,k\}$, X^* might be the set of 2^k extreme points $(x_1,\ldots,x_k)$ where $x_i = x_{i*}$ or x_i^*. Let $T' = \min_{x \in X^*} T(x)$. Since $T \leq T'$, if $T' \leq t_{1-\alpha,\nu}$ accept H_0.

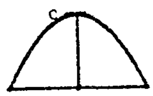
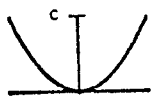
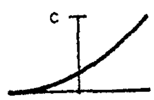
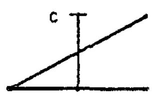
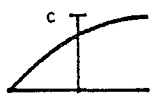
The results of a simulation study are given in Tables II and III. This study is described in Section 5. In Tables II and III the second (middle) number for each entry is the proportion of the acceptances which were determined by the shortcut method. $X^* = \{1, -1\}$ was used. These values indicate that the usefulness of this shortcut depends on the actual regression functions. But, in many cases, a large proportion of the acceptances were determined by this shortcut. In 18 out of the 62 cases in which there were some acceptances, all of the acceptances were determined by the shortcut.

4.2 Shortcut for determining if H_0 is rejected

For this shortcut to be valid, α must be no more than .5. Since α is usually $\leq .1$, this restriction is not practically important. Let m denote the number of distinct nonconstant functions in $\{f_{ij}(x): i = 1,2; j = 1,\ldots,p_i\}$. Let Z^* denote the set of 2^m points $(z_1^*, z_2^*) = ((z_{11},\ldots,z_{1p_1}), (z_{21},\ldots,z_{2p_2}))$ formed by replacing $f_{ij}(x)$ by either $\max_{x \in R} f_{ij}(x)$ or $\min_{x \in R} f_{ij}(x)$ in $(f_1(x), f_2(x))$. Note that if $f_{1r}(x) = f_{2s}(x)$ then $z_{1r} = z_{2s}$, i.e.,

TABLE II

Power of the Test and Percentage of Acceptances and Rejections by Shortcuts[1] for Selected Points $\delta(x)=f_1(x)\beta_1-f_2(x)\beta_2$ in H_0.

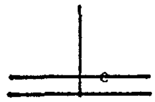
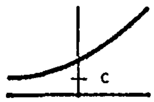
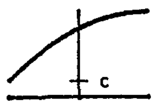
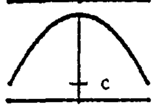
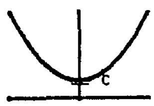
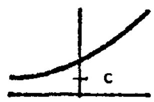
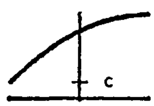
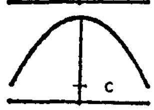
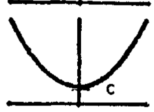
		c					
		0	1	2	5	25	1000
I		.0003	.0027	.0053	.0060	.0060	.0060
		99	100	100	100	100	100
$\delta(x)=c(1-x^2)$		100	50	75	100	100	100
II		.0003	.0080	.0287	.0483	.0510*	.0510*
		99	90	40	0	0	0
$\delta(x)=cx^2$		100	33	22	16	15	15
III		.0003	.0037	.0110	.0267	.0460	.0523*
		99	98	97	97	99	100
$\delta(x)=c(x+1)^2/4$		100	27	30	16	9	8
IV		.0003	.0067	.0237	.0493	.0523*	.0523*
		99	98	98	100	100	100
$\delta(x)=c(x+1)/2$		100	50	39	32	30	30
V		.0003	.0087	.0327	.0523*	.0523*	.0523*
		99	99	99	100	100	100
$\delta(x)=c(-x^2+2x+3)/4$		100	50	63	78	100	100

[1] First (top) entry: estimated power of the test
Second (middle) entry: percentage of acceptances detected by shortcut in Section 4.1
Third (bottom) entry: percentage of rejections detected by shortcut in Section 4.2

*Theoretically these values are $\leq .05$. These estimates are > .05 due to sampling error.

TABLE III

Power of the Test and Percentage of Acceptances and Rejections by Shortcuts[1] for Selected Points $\delta(x)=f_1(x)\beta_1-f_2(x)\beta_2$ in H_A.

		c				
		.5	1	2	3	5
I		.0053	.0510	.5197	.9323	1.000
		98	94	82	91	--
$\delta(x)=c$		38	45	65	90	100
II		.0290	.1460	.6897	.9647	1.000
		94	87	81	93	--
$\delta(x)=(x+1)^2/4+c$		33	44	65	90	100
III		.0547	.2023	.7357	.9667	1.000
		96	94	96	99	--
$\delta(x)=(-x^2+2x+3)/4+c$		51	63	81	97	100
IV		.0180	.0990	.6030	.9387	1.000
		99	99	99	100	--
$\delta(x)=-x^2+1+c$		76	78	92	100	100
V		.0560	.2327	.7967	.9907	1.000
		73	51	30	29	--
$\delta(x)=x^2+c$		21	36	59	88	100
VI		.0870	.2543	.7557	.9677	1.000
		94	92	94	99	--
$\delta(x)=(x+1)^2+c$		11	15	32	68	100
VII		.1453	.3167	.7700	.9680	1.000
		100	100	100	100	--
$\delta(x)=-x^2+2x+3+c$		79	87	96	100	100
VIII		.0250	.1107	.6077	.9387	1.000
		100	100	100	100	--
$\delta(x)=-4x^2+4+c$		100	100	100	100	100
IX		.1510	.3680	.8460	.9933	1.000
		0	0	0	0	--
$\delta(x)=4x^2+c$		17	28	57	88	100

[1] See Table II footnote.

the maximum or minimum is used in both z_1^* and z_2^*. Let $T^* = \min_{z^* \varepsilon Z^*} T(z^*)$ where

$$T(z^*) = (z_1^* b_1 - z_2^* b_2)/s \sqrt{z_1^* D_1^{-1} z_1^{*\prime} + z_2^* D_2^{-1} z_2^{*\prime}} \ .$$

If $T^* > t_{1-\alpha,\nu}$ then $T > t_{1-\alpha,\nu}$ and H_0 can be rejected. Furthermore, if $T^* = T(z^*)$ where $z^* = (f_1(x), f_2(x))$ for some $x \varepsilon R$ then, in fact $T^* = T$.

For example, suppose $f_1(x) = (1, x, x^2)$, and $f_2(x) = (1, x)$, and $R = \{x: -1 \leq x \leq 2\}$. There are two distinct nonconstant functions, x and x^2, and the $2^2 = 4$ values of (z_1^*, z_2^*) in Z^* are $((1, -1, 0), (1, -1))$, $((1, -1, 4), (1, -1))$, $((1, 2, 0), (1,2))$ and $((1, 2, 4), (1, 2))$. Points like $((1, -1, 0), (1, 2))$ where x has been replaced by its minimum in z_1^* and its maximum in z_2^* are not in Z^*.

The validity of this shortcut is based on two facts, the fact about functions which are the ratios of linear functions and square roots of positive quadratic functions mentioned in the proof of Theorem 2, Section 6, and the fact that $A = \{(f_1(x), f_2(x)): x \varepsilon R\}$ is a subset of

$$B = \{(z_1, z_2): \min_{x \varepsilon R} f_{ij}(x) \leq z_{ij} \leq \max_{x \varepsilon R} f_{ij}(x) \ \text{ and }$$
$$z_{1r} = z_{2s} \text{ if } f_{1r}(x) = f_{2s}(x)\}$$

so a minimum over A is not less than a minimum over B.

This shortcut was used in the simulation study of Section 5. In this case $R = \{x: -1 \leq x \leq 1\}$. $f_1(x) = f_2(x) = (1, x, x^2)$ so there are $m = 2$ distinct nonconstant functions. The $2^m = 4$ points in Z^* are $((1, -1, 0), (1, -1, 0))$, $((1, -1, 1), (1, -1, 1))$, $((1, 1, 0), (1, 1, 0))$ and $((1, 1, 1), (1, 1, 1))$.

In Tables II and III, the third (bottom) number for each entry is the proportion of the rejections which were detected by this shortcut. The usefulness of this shortcut is seen to depend on the actual value of the regression function but in many cases the proportion is fairly high. In 17 out of 71 cases all of the rejections were detected by this method, avoiding numerical minimization of $T(x)$.

5. POWER FUNCTION

A simulation study was conducted to investigate the power function of the test. In this study the regression functions were $f_i(x)\beta_i = \beta_{i1} + \beta_{i2}x + \beta_{i3}x^2$, $i = 1, 2$. The variance σ^2 was set equal to one. The sample sizes n_1 and n_2 were both 10 with 3 observations at each of $x = 1$ and $x = -1$ and 4 observations at $x = 0$. $R = \{x: -1 \leq x \leq 1\}$. The size of the test was fixed at $\alpha = .05$ by using $t_{.95,14} = 1.761$. The International Mathematics and Statistics Library programs GGNSM and GGCHS were used to generate the random vector $b_1 - b_2$ and the random variable s^2. A total of 3000 repetitions were used to obtain each of the estimates in Tables II and III.

The maximum probability of a Type I error takes place when $f_1(x)\beta_1 = f_2(x)\beta_2$ for one x and $f_1(x)\beta_1 - f_2(x)\beta_2$ becomes large for all other x. This can be observed in Table II where the probability of a Type I error is given for various values of β_1 and β_2 in the null hypothesis. As one proceeds across rows II, III, IV or V of Table II, $f_1(x)\beta_1 = f_2(x)\beta_2$ for one value of x ($x = -1$ for Rows III, IV and V and $x = 0$ for Row II) and $f_1(x)\beta_1 - f_2(x)\beta_2$ is becoming large for all other values of x. The probability of a Type I error increases to $\alpha = .05$ as one proceeds across any row. The estimates slightly exceed .05 in a few cases due to sampling error.

The power function of the proposed test exhibits the following monotonicity property. If (β_1, β_2) and (β_1^*, β_2^*) are two parameter vectors which satisfy

$$f_1(x)\beta_1^* - f_2(x)\beta_2^* \geq f_1(x)\beta_1 - f_2(x)\beta_2 \tag{5.1}$$

for every x with strict inequality for some x, then the power at

(β_1^*, β_2^*) is greater than the power at (β_1, β_2). This property is apparent as one proceeds across any row in Tables II or III.

The power of the test is near one only if $f_1(x)\beta_1 - f_2(x)\beta_2$ is large for all x. This is the case for the rightmost entries in Table III. In Table III, the minimum distance between the regression functions is c. The power nears one only as the minimum distance c becomes large.

The test we propose is biased in that the probability of rejecting H_0 is less than α for some (β_1, β_2) in H_A. The feature was noted by Tsutakawa and Hewett (1978) for the special model they considered and it continues to exist for the more general models we consider. This biasedness can be observed in the entries for c = .5 which are less that .05 in Table III. But as noted by Tsutakawa and Hewett (1978) for their special case, the test we propose is consistent in that, for any fixed point (β_1, β_2) in H_A, the power can be made arbitrarily near one by choosing the sample sizes sufficiently large. Although we do not feel this bias is serious, it should be noted that the power of the test we propose may be small if $f_1(x)\beta_1$ exceeds $f_2(x)\beta_2$ by only a small amount over most of R.

The power function properties we have described in this section are true in general, not just for the case of quadratic regression we considered in the simulation experiment. The proofs of these facts can be accomplished using the methods employed in proofs in Section 6.

6. SIZE AND EQUIVALENCE RESULTS

Results regarding the size of the test and the equivalence of the test to the test proposed by Tsutakawa and Hewett (1978) and Hewett and Lababidi (1980) are given in this section.

Theorem 1: Under the assumptions of our model the test has level α, i.e.,

$$\sup_{(\beta_1,\beta_2)\in H_0} P_{\beta_1,\beta_2}(T > t_{1-\alpha,\nu}) \le \alpha . \tag{6.1}$$

If in addition (2.3), (2.4), and (2.5) are true then the test has size exactly α, i.e., (6.1) is true with = replacing $\leq$.

The proof of Theorem 1 will use Lemma 1 which can be proved using standard analysis methods.

Lemma 1: Let $g_n(x)$, $n = 1, 2, \ldots$, be continuous functions on a compact set R. Suppose there exists an $x_0 \varepsilon R$ such that $g_n(x_0)$ is constant (say c) for all n. Suppose $g_n(x)$ increases to infinity as $n \to \infty$ for all $x \neq x_0$. Then

$$\lim_{n\to\infty} \min_{x\varepsilon R} g_n(x) = c. \tag{6.2}$$

Proof of Theorem 1: Fix $(\beta_1', \beta_2') \varepsilon H_0$. There is an $x_0 \varepsilon R$ such that $f_1(x_0)\beta_1' \leq f_2(x_0)\beta_2'$. Consider testing H_{0x_0}: $f_1(x_0)\beta_1 \leq f_2(x_0)\beta_2$ versus H_{Ax_0}: $f_1(x_0)\beta_1 > f_2(x_0)\beta_2$. The test which rejects H_{0x_0} if $T(x_0) > t_{1-\alpha,\nu}$ is a level α test of H_{0x_0}. Since $T \leq T(x_0)$ and $(\beta_1', \beta_2') \varepsilon H_{0x_0}$, then

$$P_{\beta_1',\beta_2'}(T > t_{1-\alpha,\nu}) \leq P_{\beta_1',\beta_2'}(T(x_0) > t_{1-\alpha,\nu}) \leq \alpha .$$

Since (β_1', β_2') was arbitrary, (6.1) is true.

Since (6.1) is true, to prove the second part of the theorem it suffices to show there exists a sequence (β_1^n, β_2^n), $n = 1,2,\ldots$, such that $(\beta_1^n, \beta_1^n) \varepsilon H_0$ for $n = 1,2,\ldots$, and

$$\lim_{n\to\infty} P_{\beta_1^n,\beta_2^n} (T > t_{1-\alpha,\nu}) \geq \alpha. \tag{6.3}$$

The estimates b_i, $i = 1,2$, can be written as $b_i = Z_i + \beta_i$ where Z_1, Z_2 and s are independent, and Z_i has a p_i-variate normal distribution with mean 0 and variance-covariance matrix $\sigma^2 D_i^{-1}$. In terms of these quantities, the statistics T and T(x) can be written as

$$T = T(Z_1, Z_2, s, \beta_1, \beta_2) = \min_{x\varepsilon R} T(x;Z_1, Z_2, s, \beta_1, \beta_2)$$

and

$$T(x;Z_1, Z_2, s, \beta_1, \beta_2)$$

$$= \frac{f_1(x)Z_1 - f_2(x)Z_2 + f_1(x)\beta_1 - f_2(x)\beta_2}{s\sqrt{e(x)}} .$$

Consider the sequence (β_1^n, β_2^n) defined by $\beta_i^n = n\beta_i^*$ where the β_i^* are the parameters identified in (2.5). For a fixed value of $z_1^* \in R^{p1}$, $z_2^* \in R^{p2}$ and $s^* > 0$, define

$$g_n(x) = T(x; z_1^*, z_2^*, s^*, \beta_1^n, \beta_2^n) .$$

The $g_n(x)$ satisfy the conditions of Lemma 1 since 1) f_{ij} are continuous, 2) $s^*\sqrt{e(x)} > 0$, 3) $f_1(x_0)\beta_1^n = f_2(x_0)\beta_2^n$ and 4) $f_1(x)\beta_1^n - f_2(x)\beta_2^n$ increases to infinity as $n \to \infty$ for all $x \neq x_0$. By Lemma 1, $\lim_{n\to\infty} T(z_1^*, z_2^*, s^*, \beta_1^n, \beta_2^n) = T(x_0; z_1^*, z_2^*, s^*, \beta_1^*, \beta_2^*)$. Since z_1^*, z_2^*, and s^*, were arbitrary, this implies that the random variables $T(Z_1, Z_2, s, \beta_1^n, \beta_2^n)$ converge to $T(x_0; Z_1, Z_2, s, \beta_1^*, \beta_2^*)$ with probability one and hence in distribution. Thus

$$\begin{aligned} \lim_{n\to\infty} P_{\beta_1^n,\beta_2^n}(T > t_{1-\alpha,\nu}) &= \lim_{n\to\infty} P(T(Z_1, Z_2, s, \beta_1^n, \beta_2^n) > t_{1-\alpha,\nu}) \\ &= P(T(x_0; Z_1, Z_2, s, \beta_1^*, \beta_2^*) > t_{1-\alpha,\nu}) \\ &= \alpha . \; || \end{aligned}$$

Conditions (2.3), (2.4), and (2.5) are satisfied in these two simple cases. These conditions are satisfied if the $f_{ij}(x)$ include the constant 1, the linear functions x_i, $i = 1,\ldots,k$, and the quadratic functions $x_i x_j$, $i = 1,\ldots,k$, $j = 1,\ldots,i$. Then β_1 and β_2 can be chosen so that $f_1(x)\beta_1 - f_2(x)\beta_2 = (x - x_0)(x - x_0)'$. Another situation in which the condition is satisfied is if

$$f_i(x)\beta_i = \beta_{i0} + \sum_{j=1}^{k} \beta_{ij}x_j$$

and $R = \{x: x_{j*} \leq x_j \leq x_j^*, j = 1,\ldots,k\}$, the model considered by Tsutakawa and Hewett (1978) and Hewett and Lababidi (1980). Then β_1 and β_2 can be chosen so that

$$f_1(x)\beta_1 - f_2(x)\beta_2 = \sum_{j=1}^{k} (x_j - x_{j*})$$

which is zero for $x = (x_{1*},\ldots,x_{k*})$ and positive for all other $x \in R$.

Theorem 2: Suppose

$$f_i(x)\beta_i = \beta_{i0} + \sum_{j=1}^{k} \beta_{ij}x_j$$

and R has the form $R = \{x:\ x_{j*} \le x_j \le x_j^*,\ j = 1,\ldots,k\}$. Consider the test which rejects H_0 if $T^* > t_{1-\alpha,\nu}$ where $T^* = \min_{x\varepsilon X^*} T(x)$ and X^* is the set of 2^k points for which x_j is either x_{j*} or x_j^*. Suppose $\alpha \le .5$. Then the tests based on T^* and T are equivalent.

Proof. For any k + 1 dimensional vectors b_1 and b_2 and $s > 0$, $T(x)$ is a linear function of $(x_1,\ldots,x_k)$ divided by the square root of a quadratic function of $(x_1,\ldots,x_k)$ which is positive for all $(x_1,\ldots,x_k) \in R^k$. Such a function has the property that $T^* = \min_{x\varepsilon X^*} T(x) \ge 0$ implies $T^* = \min_{x\varepsilon R} T(x) = T$. (This is easily proved for k = 1 and can be proved for general k by induction.) Suppose b_1, b_2 and s are such that T^* rejects H_0. Then $T^* > t_{1-\alpha,\nu} \geqq 0$, since $\alpha \leqq .5$, so $T = T^*$ and T also rejects H_0. For any b_1, b_2 and $s > 0$, $T \leqq T^*$ so if T rejects H_0, so does T^*.||

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