Available online at www.sciencedirect.com
SCIENCE DIRECT®

Nonlinear Analysis 62 (2005) 1157–1181

Nonlinear Analysis

www.elsevier.com/locate/na

Periodic solutions for scalar functional differential equations

Dan Ye[a,1], Meng Fan[a,*,2], Haiyan Wang[b]

[a]*School of Mathematics and Statistics, Key Laboratory for Vegetation Ecology of the Education Ministry of PR China, Northeast Normal University, 5268 Renmin Street, Changchun, Jilin 130024, PR China*
[b]*Department of Mathematics, University of Central Arkansas, Conway, AR 72035, USA*

Received 3 December 2003; accepted 19 March 2005

Abstract

Sufficient criteria are established for the existence of positive periodic solutions of scalar functional differential equations, which improve and generalize some related results in the literature. The approach is based on the Krasnoselskii's fixed point theorem. Numerical simulations are presented to support the analytical analysis.

MSC: 34K13; 92D25

Keywords: Scalar differential equations; Periodic solution; The Krsnoselskii fixed point theorem; Numerical simulations

1. Introduction

Let $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}_+ = [0, \infty)$. For each $x = (x_1, \dots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, the norm of x is defined as $|x| = \max_{1 \leqslant i \leqslant n} |x_i|$. $\mathbb{R}^n_+ = \{(x_1, \dots, x_n)^{\mathrm{T}} \in \mathbb{R}^n : x_i \geqslant 0, 1 \leqslant i \leqslant n\}$.

* Corresponding author. Tel.: +86 431 5099616; fax: +86 431 5098237.
E-mail addresses: mfan@nenu.edu.cn (M. Fan), hwang@uca.edu (H. Wang).

[1] Current address: School of Information Science and Engineering, Northeast University, Shenyang, Liaoning 110004, PR China.

[2] Supported by the National Natural Science Foundation of PR Chinaand the Key Project on Science and Technology of the Education Ministry of PR China.

0362-546X/$ - see front matter © 2005 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2005.03.084

In this paper, we are interested in the scalar functional differential equations of the form

$$\dot{y}(t) = -a(t)y(t) + f(t, u(t)) \tag{1.1}$$

and

$$\dot{y}(t) = a(t)y(t) - f(t, u(t)), \tag{1.2}$$

where

$$u(t) = \left(y(g_1(t)), y(g_2(t)) \ldots y(g_{n-1}(t)), \int_{-\infty}^{t} k(t-\theta)y(\theta)\,\mathrm{d}\theta \right), \tag{1.3}$$

together with the following assumptions:

(H_1) $a \in C(\mathbb{R}, \mathbb{R}_+)$ is ω-periodic i.e. $a(t) = a(t + \omega)$, such that $a(t) \not\equiv 0$, where ω is a positive constant denoting the common period of the system.
(H_2) $f \in C(\mathbb{R} \times \mathbb{R}^n_+, \mathbb{R}_+)$ is ω-periodic with respect to the first variable, i.e., $f(t + \omega, u_1 \ldots u_n) = f(t, u_1 \ldots u_n)$, such that $f(t, u) \not\equiv 0$.
(H_3) The delay kernel $k : \mathbb{R}_+ \to \mathbb{R}_+$ is integrable and is normalized such that $\int_0^{+\infty} k(r)\,\mathrm{d}r = 1$; $g_i : \mathbb{R} \to \mathbb{R}$ such that $g(t) \leqslant t$.

Models of form (1.1) and (1.2) have been proposed for population dynamics (single species growth models), physiological processes (such as production of blood cells, respiration and cardiac arrhythmias) and other practical problems. (1.1) and (1.2) are very general and incorporate many famous mathematical models extensively studied in the literature (see e.g., [1,2,4–9,11–17,19–22]).

Recently, scalar differential equations of form (1.1) and (1.2) attract much attention from both mathematicians and mathematical biologists [1,2,4–9,11–17,19–22]. Many authors devote themselves to exploring the existence of periodic solutions of some special cases of (1.1) and (1.2), see, e.g., [1,8,9,19,20]. Though some excellent results have been achieved, such equations and the corresponding existence problems are still very far from being well and systematically studied.

It is our belief that one of the most interesting and important problems in the study of dynamical models and their applications is to describe the nature of the solutions for a large range of parameters involved. From a numerical point of view, the existence of a periodic solution of the appropriate scheme must also be studied. The usual approach to fulfill such requirements is to have a set of test equations or models for which explicit analytical conditions can be given. The next step is to characterized the numerical methods which show the same existence result under the same conditions when applied to the test equations.

In this paper, we provide a unified approach to the systematic study of existence of positive periodic solutions of (1.1) and (1.2) under general conditions and then apply the obtained results to some well-known models in population dynamics, hematopoiesis, etc. both theoretically and numerically. The approach is based on the Krasnoselskii fixed point theorem, which has been applied widely in the literature and has been proved to be powerful and effective in dealing with the existence problems.

For the sake of convenience and simplicity, we now introduce below some notations that apply throughout this paper.

Define

$$\max f_0 = \lim_{|u|\uparrow 0} \max_{t\in[0,\omega]} \frac{f(t,u)}{|u|}, \quad \min f_0 = \lim_{\substack{|u|\uparrow 0 \\ u_j \geqslant \delta|u| \\ 0\leqslant j\leqslant n}} \min_{t\in[0,\omega]} \frac{f(t,u)}{|u|},$$

$$\max f_\infty = \lim_{|u|\uparrow +\infty} \max_{t\in[0,\omega]} \frac{f(t,u)}{|u|}, \quad \min f_\infty = \lim_{\substack{|u|\uparrow +\infty \\ u_j \geqslant \delta|u| \\ 0\leqslant j\leqslant n}} \min_{t\in[0,\omega]} \frac{f(t,u)}{|u|},$$

$$G(t,s) = \frac{\exp\{\int_t^s a(\xi)\,\mathrm{d}\xi\}}{\exp\{\int_0^\omega a(\xi)\,\mathrm{d}\xi\} - 1}, \ t,s \in \mathbb{R}; \quad \delta = \exp\left\{-\int_0^\omega a(s)\,\mathrm{d}s\right\}$$

and the number set $C := \{\max f_0, \min f_0, \max f_\infty, \min f_\infty\}$.

The tree of this paper is as follows:

1 Introduction

2 Existence of positive periodic solutions of (1.1)

- 2.1 Preliminary
- 2.2 Case 1 : $C \subset \{0, +\infty\}$
- 2.3 Case 2 : $C \cap \{0, +\infty\} = \emptyset$
- 2.4 Case 3 : $C \cap \{0, +\infty\} \neq \emptyset$, but $C \not\subset \{0, +\infty\}$

3 Existence of positive periodic solutions of (1.2)

4 Examples and numerical simulations

5 Conclusive remarks

2. Existence of periodic solution of (1.1)

We establish the existence of positive periodic solutions of (1.1) and (1.2) by applying the Krasnoselskii fixed point theorem on cones. We shall first make some preparations and list below a few preliminary results.

2.1. Preliminary

First, let us introduce the Krasnoselskii fixed point theorem which will come into play later on.

Definition 2.1. Let X be a Banach space and E be a closed, nonempty subset of X. E is said to be a cone if

(i) $\alpha u + \beta v \in$ for all $u, v \in E$ and all $\alpha, \beta > 0$,
(ii) $u, -u \in E$ imply $u = 0$.

Lemma 2.1 (*Krasnoselskii [10], fixed point theorem*). *Let X be a Banach space, and let E $\subset X$ be a cone in X. Assume Ω_1, Ω_2, are open subsets of X with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$ and let*

$$\Phi : E \cap (\bar{\Omega}_2 \backslash \Omega_1) \to E$$

be a completely continuous operator such that either

(i) $\|\Phi y\| \geqslant \|y\|$ *for any* $y \in E \cap \partial\Omega_1$ *and* $\|\Phi y\| \leqslant \|y\|$ *for any* $y \in E \cap \partial\Omega_2$

or

(ii) $\|\Phi y\| \leqslant \|y\|$ *for any* $y \in E \cap \partial\Omega_1$ *and* $\|\Phi y\| \geqslant \|y\|$ *for any* $y \in E \cap \partial\Omega_2$.

Then Φ has a fixed point in $E \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

By the definition of $G(t, s)$, one can easily reach the following claims.

Lemma 2.2. *$G(t, s)$ satisfies*

(i) $G(t, s) = G(t + \omega, s + \omega)$ *for any* $t, s \in \mathbb{R}$,
(ii) $A = \delta/(1 - \delta) = G(t, t) \leqslant G(t, s) \leqslant G(t, t + \omega) = 1/(1 - \delta) = B$, *for* $s \in [t, t + \omega]$,
(iii) $\delta \leqslant G(t, s)/G(t, t + \omega) \leqslant 1$, *for* $s \in [t, t + \omega]$.

Lemma 2.3. *$y(t)$ is an ω periodic solution of Eq.* (1.1) *if and only if it is also an ω periodic solution of the following integral equation*:

$$y(t) = \int_t^{t+\omega} G(t, s) f \left(s, \left(y(g_1(s)), y(g_2(s)) \ldots y(g_{n-1}(s)), \int_{-\infty}^{s} k(s - \theta) y(\theta)\, \mathrm{d}\theta \right) \right) \mathrm{d}s. \tag{2.1}$$

Let

$$X = \{y(t) : y(t) \in C(\mathbb{R}, \mathbb{R}), y(t + \omega) = y(t)\}$$

and define

$$\|y\| = \sup_{t \in [0, \omega]} \{|y(t)| : y \in X\},$$

then X is a Banach space when endowed with the norm $\|\cdot\|$. Moreover, from (1.3), we know $|u| \leqslant \|y\|$. Define the operator Φ on X as

$$(\Phi y)(t) = \int_t^{t+\omega} G(t, s) f \left(s, \left(y(g_1(s)), y(g_2(s)) \ldots y(g_{n-1}(s)), \int_{-\infty}^{s} k(s - \theta) y(\theta)\, \mathrm{d}\theta \right) \right) \mathrm{d}s \tag{2.2}$$

for $y \in X$ and $t \in \mathbb{R}$. It is trivial to how that Φ is a completely continuous operator on X. It is clear that $y = y(t)$ is an ω-periodic solution of (1.1) whenever y is a fixed point of Φ, i.e., $y = \Phi y$.

Define

$$E = \{y \in X : y(t) \geqslant 0 \text{ and } y(t) \geqslant \delta \|y\|\},$$

one may readily verify that E is a cone.

Lemma 2.4. *$\Phi : E \to E$ is well-defined.*

Proof. In view of Lemma 2.2, for each $y \in E$, we have $(\Phi y)(t)$ is continuous in t and $(\Phi y)(t + \omega) = (\Phi y)(t)$. Moreover,

$$\|\Phi y\| \leqslant \frac{1}{1-\delta} \int_0^\omega f\left(s, \left(y(g_1(s)), y(g_2(s)) \ldots y(g_{n-1}(s)), \int_{-\infty}^{s} k(s-\theta) y(\theta)\, \mathrm{d}\theta\right)\right) \mathrm{d}s$$

and

$$(\Phi y)(t) \geqslant \frac{\delta}{1-\delta} \int_0^\omega f\left(s, \left(y(g_1(s)), y(g_2(s)) \ldots y(g_{n-1}(s)), \int_{-\infty}^{s} k(s-\theta) y(\theta)\, \mathrm{d}\theta\right)\right) \mathrm{d}s \geqslant \delta \|\Phi y\|,$$

therefore $\Phi y \in E$. The proof is complete. □

2.2. Case 1: $C \subset \{0, +\infty\}$.

Theorem 2.1. *Assume that*

$$(P_1) \quad \min f_0 = \infty \text{ and } \max f_\infty = 0.$$

Then (1.1) *has at least one positive ω-periodic solution.*

Proof. First, in view of $\min f_0 = \infty$, for any $M_1 > 1/(A\delta\omega)$, there exists a $\rho_0 > 0$ such that

$$f(t, u) \geqslant M_1 |u| \quad \text{for } u_j \geqslant \delta |u|, \quad 0 \leqslant j \leqslant n, \quad |u| \leqslant \rho_0. \tag{2.3}$$

Set

$$\Omega_1 = \{y \in X : \|y\| < \rho_0\}.$$

Then, for any $y \in E \cap \partial\Omega_1$, we have $y(t) \geqslant \delta\rho_0$. Thus, for $u(t)$ defined by (1.3), we have

$$\begin{aligned}
&u_j(t) = y(g_j(t)) \geqslant \delta\|y\| \geqslant \delta|u(t)|, \quad j = 1, 2 \dots n-1,\\
&u_n(t) = \int_{-\infty}^{t} k(t-\theta)y(\theta)\,\mathrm{d}\theta \geqslant \delta\|y\| \geqslant \delta|u(t)|,\\
&\delta\rho_0 \leqslant |u(t)| = \max_{0 \leqslant j \leqslant n-1} \left\{ y(g_j(t)), \int_{-\infty}^{t} k(t-\theta)y(\theta)\,\mathrm{d}\theta \right\} \leqslant \|y\| = \rho_0.
\end{aligned}$$

By (2.2), (2.3) and Lemma 2.2, we get

$$\begin{aligned}
(\Phi y)(t) &\geqslant A \int_{t}^{t+\omega} f\left(s, \left(y(g_1(s)), y(g_2(s)) \dots y(g_{n-1}(s)),\right.\right.\\
&\qquad \left.\left.\int_{-\infty}^{s} k(s-\theta)y(\theta)\,\mathrm{d}\theta\right)\right)\,\mathrm{d}s\\
&\geqslant AM_1\omega|u| \geqslant AM_1\delta\rho_0\omega > \rho_0 = \|y\|,
\end{aligned}$$

which implies that $\|\Phi y\| \geqslant \|y\|$ for any $y \in E \cap \partial\Omega_1$.

On the other hand, since $\max f_\infty = 0$, for any $0 < \varepsilon < 1/(2B\omega)$, there exists an $N_1 > \rho_0$ such that

$$0 \leqslant f(t,u) \leqslant \varepsilon|u| \quad \text{for } |u| \geqslant N_1.$$

Let

$$\rho_1 > 2N_1 + 2B\omega \max_{\substack{t\in[0,\omega]\\ |u| \leqslant N_1\\ u \in \mathbb{R}_0^{+n}}} f(t,u)$$

and $\Omega_2 := \{y \in X : \|y\| < \rho_1\}$. Then for any $y \in E \cap \partial\Omega_2$, we have $\|y\| = \rho_1$, and hence

$$\begin{aligned}
(\Phi y)(t) &\leqslant B \int_{t}^{t+\omega} f\left(s, \left(y(g_1(s)), y(g_2(s)) \dots y(g_{n-1}(s)),\right.\right.\\
&\qquad \left.\left.\int_{-\infty}^{s} k(s-\theta)y(\theta)\,\mathrm{d}\theta\right)\right)\,\mathrm{d}s\\
&= B \int_{I_1} f\left(s, \left(y(g_1(s)), y(g_2(s)) \dots y(g_{n-1}(s)),\right.\right.\\
&\qquad \left.\left.\int_{-\infty}^{s} k(s-\theta)y(\theta)\,\mathrm{d}\theta\right)\right)\,\mathrm{d}s\\
&\quad + B \int_{I_2} f\left(s, \left(y(g_1(s)), y(g_2(s)) \dots y(g_{n-1}(s)),\right.\right.\\
&\qquad \left.\left.\int_{-\infty}^{s} k(s-\theta)y(\theta)\,\mathrm{d}\theta\right)\right)\,\mathrm{d}s\\
&\leqslant \frac{\rho_1}{2} + B\omega\varepsilon\|y\| \leqslant \frac{\rho_1}{2} + \frac{\|y\|}{2} = \|y\|,
\end{aligned}$$

where

$$I_1 = \{s \in [0, \omega], |u(s)| \leqslant N_1\}, \quad I_2 = \{s \in [0, \omega], |u(s)| > N_1\}.$$

This implies that $\|\Phi y\| \leqslant \|y\|$ for any $y \in E \cap \partial\Omega_2$. By now, we have shared all the requirements in Lemma 2.1, then Φ has a fixed point $y \in E \cap (\bar{\Omega}_2 \backslash \Omega_1)$. Clearly, we have $\rho_0 \leqslant \|y\| \leqslant \rho_1$ and $y(t) \geqslant \delta\|y\| > 0$, which means that $y(t)$ is a positive ω-periodic solution of (2.1). Hence, by Lemma 2.3, $y(t)$ is a positive ω-periodic solution of (1.1). The proof is complete. □

Theorem 2.2. *Assume that*

$$(P_2) \max f_0 = 0 \quad \textit{and} \quad \min f_\infty = \infty.$$

Then (1.1) *has at least one positive ω-periodic solution*.

Proof. Since $\max f_0 = 0$, for any $0 < \varepsilon < 1/(B\omega)$, there exists $\rho_2 > 0$ such that

$$f(t, u) \leqslant \varepsilon|u| \quad \text{for } |u| \leqslant \rho_2. \tag{2.4}$$

Define $\Omega_1 = \{y \in X : \|y\| < \rho_2\}$. Then, for any $y \in E \cap \partial\Omega_1$, we have $\|y\| = \rho_2$ and $y(t) \geqslant \delta\rho_2$. Moreover, by (1.3), we have $|u| \leqslant \|y\|$. From (2.2), (2.4) and Lemma 2.2, it follows that

$$\begin{aligned}(\Phi y)(t) &\leqslant B \int_t^{t+\omega} f\left(s, \left(y(g_1(s)), y(g_2(s)) \dots y(g_{n-1}(s)), \right.\right.\\ &\qquad \left.\left.\int_{-\infty}^{s} k(s-\theta) y(\theta)\, \mathrm{d}\theta\right)\right) \mathrm{d}s \\ &\leqslant B\varepsilon|u|\omega \leqslant B\varepsilon\|y\|\omega \leqslant \|y\|,\end{aligned}$$

which implies that $\|\Phi y\| \leqslant \|y\|$ for any $y \in E \cap \partial\Omega_1$.

Next, since $\min f_\infty = \infty$, for any $M_2 > 1/(A\delta\omega)$, there exists a $\rho_3 > \rho_2/\delta$ such that

$$f(t, u) \geqslant M_2|u| \quad \text{for } u_j \geqslant \delta|u|, \quad 0 \leqslant j \leqslant n \quad \text{and} \quad |u| \geqslant \delta\rho_3. \tag{2.5}$$

Define $\Omega_2 = \{y \in X : \|y\| < \rho_3\}$, then, for any $y \in E \cap \partial\Omega_2$, we have $\|y\| = \rho_3$ and $y(t) \geqslant \delta\|y\| = \delta\rho_3$.

Consequently,

$$\begin{aligned}u_j(t) &= y(g_j(t)) \geqslant \delta\|y\| = \delta\rho_3 \geqslant \delta|u(t)| \quad j = 1 \dots n-1,\\ u_n(t) &= \int_{-\infty}^{t} k(t-\theta) y(\theta)\, \mathrm{d}\theta \geqslant \delta\|y\| = \delta\rho_3 \geqslant \delta|u(t)|,\\ |u(t)| &= \max_{0 \leqslant j \leqslant n-1} \left\{ y(g_j(t)), \int_{-\infty}^{t} k(t-\theta) y(\theta)\, \mathrm{d}\theta \right\} \geqslant \delta\rho_3,\end{aligned}$$

where $u(t)$ is defined in (1.3). From (2.2), (2.5) and Lemma 2.2, it follows that

$$\begin{aligned}(\Phi y)(t) &\geqslant A\int_t^{t+\omega} f\left(s,\left(y(g_1(s)),\, y(g_2(s))\dots y(g_{n-1}(s)),\right.\right.\\ &\qquad \left.\left.\int_{-\infty}^{s} k(s-\theta)y(\theta)\,\mathrm{d}\theta\right)\right)\,\mathrm{d}s\\ &\geqslant AM_2\delta\|y\|\omega > \|y\|,\end{aligned}$$

which implies that $\|\Phi y\| > \|y\|$ for any $y \in E\cap\partial\Omega_2$. Therefore, by Lemma 2.1, it follows that Φ has a fixed point $y \in E\cap(\bar{\Omega}_2\backslash\Omega_1)$. Obviously, $\rho_2 \leqslant \|y\| \leqslant \rho_3$ and $y(t) \geqslant \delta\|y\| > \delta\rho_2 > 0$, that is $y(t)$ is a positive ω-periodic solution of (1.1). □

Theorem 2.3. *Assume that*

(P_3) $\max f_0 = \max f_\infty = 0$,

(P_4) *There exists a* $r_1 > 0$ *such that* $f(t,u) \geqslant r_1/A\omega$ *for* $|u| \in [\delta r_1, r_1]$.

Then (1.1) *has at least two positive* ω*-periodic solutions* y_1 *and* y_2 *such that* $0 < \|y_1\| < r_1 < \|y_2\|$.

Proof. First, since $\max f_0 = 0$, for any $0 < \varepsilon < 1/(B\omega)$, there exists $\rho_4 < r_1$ such that

$$f(t,u) \leqslant \varepsilon|u| \quad \text{for } |u| < \rho_4. \tag{2.6}$$

Let $\Omega_{\rho_4} = \{y \in X : \|y\| < \rho_4\}$. Then for any $y \in E\cap\partial\Omega_{\rho_4}$, we have $\|y\| = \rho_4$. From (2.2) and (2.6), we get

$$\begin{aligned}(\Phi y)(t) &\leqslant B\int_t^{t+\omega} f\left(s,\left(y(g_1(s)),\, y(g_2(s))\dots y(g_{n-1}(s)),\right.\right.\\ &\qquad \left.\left.\int_{-\infty}^{s} k(s-\theta)y(\theta)\,\mathrm{d}\theta\right)\right)\,\mathrm{d}s\\ &\leqslant B\varepsilon\rho_4\omega \leqslant \rho_4 = \|y\|,\end{aligned}$$

which implies $\|\Phi y\| < \|y\|$ for any $y \in E\cap\partial\Omega_{\rho_4}$.

Second, in view of $\max f_\infty = 0$, for any $0 < \varepsilon\omega < 1/(2B\omega)$, there exists an $N_2 > r_1$ such that

$$f(t,u) \leqslant \varepsilon|u| \quad \text{for } |u| \geqslant N_2. \tag{2.7}$$

Let $\Omega_{\rho_5} = \{y \in X : \|y\| < \rho_5\}$, where

$$\rho_5 > 2N_2 + 2B\omega \max_{\substack{t\in[0,\omega]\\ |u|\leqslant N_2\\ u\in\mathbb{R}_0^{+n}}} f(t,u), \tag{2.8}$$

then for any $y \in E \cap \partial\Omega_{\rho_5}$, from (2.2), (2.7) and (2.8), we get

$$\begin{aligned}(\Phi y)(t) &\leqslant B\int_t^{t+\omega} f\Bigg(s, \Bigg(y(g_1(s)), y(g_2(s))\dots y(g_{n-1}(s)),\\ &\qquad \int_{-\infty}^{s} k(s-\theta)y(\theta)\,\mathrm{d}\theta\Bigg)\Bigg)\,\mathrm{d}s\\ &= B\int_{I_1} f\Bigg(s, \Bigg(y(g_1(s)), y(g_2(s))\dots y(g_{n-1}(s)),\\ &\qquad \int_{-\infty}^{s} k(s-\theta)y(\theta)\,\mathrm{d}\theta\Bigg)\Bigg)\,\mathrm{d}s\\ &\quad + B\int_{I_2} f\Bigg(s, \Bigg(y(g_1(s)), y(g_2(s))\dots y(g_{n-1}(s)),\\ &\qquad \int_{-\infty}^{s} k(s-\theta)y(\theta)\,\mathrm{d}\theta\Bigg)\Bigg)\,\mathrm{d}s\\ &\leqslant \frac{\rho_5}{2} + \frac{\|y\|}{2} = \|y\|,\end{aligned}$$

where

$$I_1 = \{s \in [0,\omega], |u(s)| \leqslant N_2\}, \quad I_2 = \{s \in [0,\omega], |u(s)| > N_2\}.$$

Whence $\|\Phi y\| < \|y\|$ for any $y \in E \cap \partial\Omega_{\rho_5}$. Finally, set $\Omega_{r_1} = \{y \in X : \|y\| < r_1\}$. Then, for any $y \in E \cap \partial\Omega_{r_1}$, we have $y(t) \geqslant \delta\|y\| = \delta r_1$. Consequently,

$$\begin{aligned}&u_j(t) = y(g_j(t)) \geqslant \delta\|y\| = \delta r_1 \quad j = 1\dots n-1,\\ &u_n(t) = \int_{-\infty}^{t} k(t-\theta)y(\theta)\,\mathrm{d}\theta \geqslant \delta\|y\| = \delta r_1,\end{aligned}$$

then

$$|u(t)| = \max_{0\leqslant j\leqslant n-1}\left\{y(g_j(t)), \int_{-\infty}^{t} k(t-\theta)y(\theta)\,\mathrm{d}\theta\right\} \geqslant \delta r_1$$

and from (1.3), we know $|u| \leqslant \|y\|$. Therefore, from (2.2) and (P_4), one has

$$\begin{aligned}(\Phi y)(t) &\geqslant A\int_t^{t+\omega} f\Bigg(s, \Bigg(y(g_1(s)), y(g_2(s))\dots y(g_{n-1}(s)),\\ &\qquad \int_{-\infty}^{s} k(s-\theta)y(\theta)\,\mathrm{d}\theta\Bigg)\Bigg)\,\mathrm{d}s\\ &\geqslant A\frac{r_1}{A\omega}\omega = r_1 = \|y\|,\end{aligned}$$

which yields $\|\Phi y\| \geqslant \|y\|$ for any $y \in E \cap \partial\Omega_{r_1}$. Therefore, by Lemma 2.1, Φ has a fixed point y_1 in $\bar{\Omega}_{r_1}\backslash\Omega_{\rho_4}$, and a fixed point y_2 in $\bar{\Omega}_{\rho_5}\backslash\Omega_{r_1}$. One can easily see that both y_1 and y_2 are positive ω-periodic solutions of (1.1) and $0 < \|y_1\| < r_1 < \|y_2\|$. The proof is complete. □

Theorem 2.4. *Assume that*

(P_5) $\min f_0 = \min f_\infty = \infty$.

(P_6) *There exists a* $r_2 > 0$ *such that* $f(t,u) \leqslant r_2/B\omega$ *for* $|u| \leqslant r_2$.

Then (1.1) *has at least two positive* ω*-periodic solutions* y_1 *and* y_2 *such that* $0 < \|y_1\| < r_2 < \|y_2\|$.

Proof. First, in view of $\min f_0 = \infty$, then for any $M_3 > 1/(A\delta\omega)$, there exists $\rho_6 < r_2$ such that

$$f(t,u) \geqslant M_3|u| \quad \text{for } |u| < \rho_6, \ \ u_j \geqslant \delta|u| \ \ (j = 1, 2 \ldots n). \tag{2.9}$$

Set $\Omega_{\rho_6} = \{y \in X : \|y\| < \rho_6\}$. Then for any $y \in E \cap \partial\Omega_{\rho_6}$, one has $\|y\| = \rho_6$, $y(t) \geqslant \delta\|y(t)\| = \delta\rho_6$. Hence, for such y, one has

$$\begin{aligned}
&u_j(t) = y(g_j(t)) \geqslant \delta\|y\| \geqslant \delta|u(t)|, \quad j = 1 \ldots n-1,\\
&u_n(t) = \int_{-\infty}^{t} k(t-\theta)y(\theta)\,\mathrm{d}\theta \geqslant \delta\|y\| \geqslant \delta|u(t)|,\\
&\delta\rho_6 \leqslant |u(t)| = \max_{0 \leqslant j \leqslant n-1} \left\{ y(g_j(t)), \int_{-\infty}^{t} k(t-\theta)y(\theta)\,\mathrm{d}\theta \right\} \leqslant \|y\| = \rho_6.
\end{aligned}$$

Therefore, from (2.2), (2.9) and Lemma 2.2, we get

$$\begin{aligned}
(\Phi y)(t) &\geqslant A \int_t^{t+\omega} f\Bigg(s, \Bigg(y(g_1(s)), y(g_2(s)) \ldots y(g_{n-1}(s)),\\
&\qquad \int_{-\infty}^{s} k(s-\theta)y(\theta)\,\mathrm{d}\theta\Bigg)\Bigg)\,\mathrm{d}s\\
&\geqslant AM_3\delta\|y\|\omega \geqslant \|y\|,
\end{aligned}$$

which yields $\|\Phi y\| \geqslant \|y\|$ for any $y \in E \cap \partial\Omega_{\rho_6}$.

In addition, since $\min f_\infty = \infty$, then for any $M_4 > 1/(A\delta\omega)$, there exists a $\rho_7 > r_2$ such that

$$f(t,u) \geqslant M_4|u| \quad \text{for } |u| > \delta\rho_7, \ \ u_j \geqslant \delta|u| \ \ (j = 1, 2 \ldots n). \tag{2.10}$$

Set $\Omega_{\rho_7} = \{y \in X : \|y\| < \rho_7\}$, then for any $y \in E \cap \partial\Omega_{\rho_7}$, we have $y(t) \geqslant \delta\|y\| = \delta\rho_7$, and furthermore

$$\begin{aligned}
&u_j(t) = y(g_j(t)) \geqslant \delta\|y\| \geqslant \delta|u(t)|, \quad j = 1 \ldots n-1,\\
&u_n(t) = \int_{-\infty}^{t} k(t-\theta)y(\theta)\,\mathrm{d}\theta \geqslant \delta\|y\| \geqslant \delta|u(t)|,\\
&|u(t)| = \max_{0 \leqslant j \leqslant n-1} \left\{ y(g_j(t)), \int_{-\infty}^{t} k(t-\theta)y(\theta)\,\mathrm{d}\theta \right\} \geqslant \delta\|y\| = \delta\rho_7.
\end{aligned}$$

From (2.2), (2.10) and Lemma 2.2, one has

$$\begin{aligned}(\Phi y)(t) &\geqslant A \int_t^{t+\omega} f\left(s, \left(y(g_1(s)), y(g_2(s)) \ldots y(g_{n-1}(s)),\right.\right.\\ &\qquad \left.\left.\int_{-\infty}^{s} k(s-\theta) y(\theta)\, \mathrm{d}\theta\right)\right) \mathrm{d}s\\ &\geqslant A M_4 \delta \rho_7 \omega \geqslant \|y\|,\end{aligned}$$

which yields $\|\Phi y\| \geqslant \|y\|$ for any $y \in E \cap \partial\Omega_{\rho_7}$.

Finally, let $\Omega_{r_2} = \{y \in X : \|y\| < r_2\}$. For any $y \in E \cap \partial\Omega_{r_2}$, we have $\|y\| = r_2$ and hence $|u(t)| \leqslant r_2$. (P_6) and Lemma 2.2 tell us that

$$\begin{aligned}(\Phi y)(t) &\leqslant B \int_t^{t+\omega} f\left(s, \left(y(g_1(s)), y(g_2(s)) \ldots y(g_{n-1}(s)),\right.\right.\\ &\qquad \left.\left.\int_{-\infty}^{s} k(s-\theta) y(\theta)\, \mathrm{d}\theta\right)\right) \mathrm{d}s\\ &\leqslant B \frac{r_2}{B\omega} \omega = r_2 = \|y\|,\end{aligned}$$

which yields $\|\Phi y\| \leqslant \|y\|$ for any $y \in E \cap \partial\Omega_{r_2}$. Therefore, by Lemma 2.1, Φ has a fixed point y_1 in $\bar{\Omega}_{r_2} \backslash \Omega_{\rho_6}$ and a fixed point y_2 in $\bar{\Omega}_{\rho_7} \backslash \Omega_{r_2}$, which are positive ω-periodic solutions of (1.1) and $0 < \|y_1\| < r_2 < \|y_2\|$. The proof is complete. □

2.3. *Case 2: $C \cap \{0, +\infty\} = \emptyset$.*

In this subsection, we discuss the existence for the positive periodic solution of (1.1) under the assumption $C \cap \{0, +\infty\} = \emptyset$. To achieve this, first, we prefer to establish a more general criterion, which plays an important role in the proof of the following theorems in this subsection.

Theorem 2.5. *Assume that* (P_4) *and* (P_6) *hold. Then* (1.1) *has at least one positive ω-periodic solution y with $\|y\|$ lying between r_2 and r_1, which are defined in* (P_4) *and* (P_6), *respectively*.

Proof. Without loss of generality, we may assume that $r_2 < r_1$. Let $\Omega_{r_2} = \{y \in X : \|y\| < r_2\}$. Then for any $y \in E \cap \partial\Omega_{r_2}$, from (2.2) and (P_6), we have

$$\begin{aligned}(\Phi y)(t) &\leqslant B \int_t^{t+\omega} f\left(s, \left(y(g_1(s)), y(g_2(s)) \ldots y(g_{n-1}(s)),\right.\right.\\ &\qquad \left.\left.\int_{-\infty}^{s} k(s-\theta) y(\theta)\, \mathrm{d}\theta\right)\right) \mathrm{d}s\\ &\leqslant B \frac{r_2}{B\omega} \omega = r_2 = \|y\|,\end{aligned}$$

which leads to $\|\Phi y\| \leqslant \|y\|$ for any $y \in E \cap \partial\Omega_{r_2}$.

Now, we set $\Omega_{r_1}=\{y\in X:\|y\|<r_1\}$. For any $y\in E\cap\partial\Omega_{r_1}$, we have $y(t)\geqslant\delta\|y\|=\delta r_1$. Then

$$\delta r_1=\delta\|y\|\leqslant|u(t)|=\max_{0\leqslant j\leqslant n-1}\left\{y(g_j(t)),\int_{-\infty}^{t}k(t-\theta)y(\theta)\,\mathrm{d}\theta\right\}$$
$$\leqslant\|y\|=r_1.$$

From (2.2) and (P_4), we get

$$(\Phi y)(t)\geqslant A\int_t^{t+\omega}f\left(s,\left(y(g_1(s)),y(g_2(s))\ldots y(g_{n-1}(s)),\right.\right.$$
$$\left.\left.\int_{-\infty}^{s}k(s-\theta)y(\theta)\,\mathrm{d}\theta\right)\right)\mathrm{d}s$$
$$\geqslant A\frac{r_1}{A\omega}\omega=r_1=\|y\|,$$

which yields $\|\Phi y\|\geqslant\|y\|$ for any $y\in E\cap\partial\Omega_{r_1}$. Therefore, the claim follows from Lemma 2.1 directly. The proof is complete. □

Theorem 2.6. *Assume that*

$$(P_7)\quad \max f_0=\alpha_1\in\left(0,\frac{1}{B\omega}\right),$$

$$(P_8)\min f_\infty=\beta_1\in\left(\frac{1}{A\delta\omega},\infty\right).$$

Then (1.1) *has at least one positive* ω*-periodic solution*.

Proof. First, since $\max f_0=\alpha_1\in(0,\frac{1}{B\omega})$ for $\varepsilon=\frac{1}{B\omega}-\alpha_1>0$, there exists a sufficiently small $r_2>0$ such that

$$\max_{t\in[0,\omega]}\frac{f(t,u)}{|u|}\leqslant\alpha_1+\varepsilon=\frac{1}{B\omega}\quad\text{for } |u|\leqslant r_2,$$

which yields

$$f(t,u)\leqslant\frac{1}{B\omega}|u|\leqslant\frac{1}{B\omega}r_2\quad\text{for } |u|\leqslant r_2,\ \ t\in[0,\omega].$$

Hence, the condition (P_6) is satisfied.

Since $\min f_\infty=\beta_1\in[\frac{1}{A\delta\omega},\infty)$ for $\varepsilon=\beta_1-\frac{1}{A\delta\omega}>0$, there exists a sufficiently large $r_1>0$ such that

$$\min_{t\in[0,\omega]}\frac{f(t,u)}{|u|}\geqslant\beta_1-\varepsilon=\frac{1}{A\delta\omega}\quad\text{for } |u|\geqslant\delta r_1,\ \ u_j\geqslant\delta|u|,\ \ j=1,2\ldots n.$$

Thus, when $|u|\in[\delta r_1,r_1]$, $t\in[0,\omega]$, we have

$$f(t,u)\geqslant\frac{1}{A\delta\omega}\delta r_1=\frac{r_1}{A\omega},$$

which implies the condition (P_4) hold. By Theorem 2.3.1, we complete the proof. □

Theorem 2.7. *Assume that*

$$(P_9)\ \min f_0 = \alpha_2 \in \left(\frac{1}{A\delta\omega}, \infty\right),$$

$$(P_{10})\ \max f_\infty = \beta_2 \in \left(0, \frac{1}{B\omega}\right).$$

Then (1.1) *has at least one positive* ω*-periodic solution*.

Proof. Since $\min f_0 = \alpha_2 \in (\frac{1}{A\delta\omega}, \infty)$ for $\varepsilon = \alpha_2 - \frac{1}{A\delta\omega} > 0$, there exists a sufficiently small $r_1 > 0$ such that

$$\min_{t\in[0,\omega]} \frac{f(t,u)}{|u|} \geqslant \alpha_2 - \varepsilon = \frac{1}{A\delta\omega} \quad \text{for } 0 \leqslant |u| \leqslant r_1 \ \ u_j \geqslant \delta|u|, \quad j = 1, 2 \ldots n.$$

Thus, when $|u| \in [\delta r_1, r_1]$, $t \in [0, \omega]$, one has

$$f(t,u) \geqslant \frac{1}{A\delta\omega}\delta r_1 = \frac{r_1}{A\omega},$$

which yields the condition (P_4) hold.

In view of $\max f_\infty = \beta_2 \in (0, 1/B\omega)$ for $\varepsilon = 1/B\omega - \beta_2 > 0$, there exists a sufficiently large r such that

$$\max_{t\in[0,\omega]} \frac{f(t,u)}{|u|} \leqslant \beta_2 + \varepsilon = 1/B\omega \quad \text{for } |u| > r. \tag{2.11}$$

In the following, we will show that (P_6) holds and the discussion is divided into two cases.

Case (i): Suppose that $\max_{t\in[0,\omega]} f(t,u)$ is unbounded, then there exists $u^* \in \mathbb{R}^n_+$, $|u^*| = r_2 > r$ and $t_0 \in [0, \omega]$ such that

$$f(t,u) \leqslant g(t_0, u^*) \quad \text{for } 0 < |u| \leqslant |u^*| = r_2. \tag{2.12}$$

Since $|u^*| = r_2 > r$, then from (2.11), (2.12), we obtain

$$f(t,u) \leqslant g(t_0, u^*) \leqslant \frac{1-\delta}{\omega}|u^*| = \frac{r_2(1-\delta)}{\delta\omega} = \frac{r_2}{B\omega} \quad \text{for } 0 < |u| \leqslant r_2, \quad t \in [0, \omega],$$

which yields the condition (P_6) hold.

Case (ii): Suppose that $\max_{t\in[0,\omega]} f(t,u)$ is bounded, say

$$f(t,u) \leqslant M \quad \text{for } (t,u) \in [0,\omega] \times \mathbb{R}^n_+. \tag{2.13}$$

In this case, taking sufficiently large $r_2 \geqslant M[\delta\omega/(1-\delta)]$, then from (2.13), we have

$$f(t,u) \leqslant M \leqslant \frac{r_2(1-\delta)}{\delta\omega} = \frac{r_2}{B\omega} \quad \text{for } 0 < |u| \leqslant r_2, \quad t \in [0, \omega],$$

which implies the condition (P_6) hold. Hence, from Theorem 2.5, the assertion follows directly. The proof is complete. □

Theorem 2.8. *Assume that* (P_6), (P_8) *and* (P_9) *hold, then* (1.1) *has at least two positive* ω*-periodic solutions* y_1 *and* y_2 *satisfying* $0 < \|y_1\| < r_2 < \|y_2\|$, *where* r_2 *is defined in* (P_6).

Proof. From (P_8) and the proof of Theorem 2.6, we know that there exists a sufficiently large $r_1 > r_2$ such that

$$f(t,u) \geqslant \frac{r_1}{A\omega} \quad \text{for } |u| \in [\delta r_1, r_1].$$

In view of (P_9) and the proof of Theorem 2.3.3, we see that there exists a sufficiently small $r_1^* \in (0, r_2)$ such that

$$f(t,u) \geqslant \frac{r_1^*}{a\omega} \quad \text{for } |u| \in [\delta r_1^*, r_1^*].$$

Therefore, from the proof of Theorem 2.3.1, we know that (1.1) has two positive ω-periodic solutions y_1 and y_2 such that $r_1^* < \|y_1\| < r_2 < \|y_2\| < r_1$. The proof is complete. □

Theorem 2.9. *Assume that* (P_4) ,(P_7) *and* (P_{10}) *hold, then* (1.1) *has at least two positive* ω*-periodic solutions* y_1 *and* y_2 *such that* $0 < \|y_1\| < r_1 < \|y_2\|$, *where* r_1 *is defined in* (P_4).

Proof. From (P_7) and the proof of Theorem 2.6, we know that there exists a sufficiently small $r_2 \in (0, r_1)$ such that

$$f(t,u) \leqslant \frac{r_2}{B\omega} \quad \text{for } |u| \leqslant r_2, \quad t \in [0, \omega].$$

In view of $(P_{10})$and the proof of Theorem 2.7, we know that there exists a sufficiently large $r_2^* > r_1$ such that

$$f(t,u) \leqslant \frac{r_2^*}{B\omega} \quad \text{for } |u| \leqslant r_2^*, \quad t \in [0, \omega].$$

Therefore, by the proof of Theorem 2.5, we see that (1.1) has two positive ω-periodic solutions y_1 and y_2 such that $r_2 < \|y_1\| < r_1 < \|y_2\| < r_2^*$. This completes the proof . □

2.4. Cases 3: $C \cap \{0, +\infty\} \neq \emptyset$*, but* $C \not\subset \{0, +\infty\}$.

Theorem 2.10. *Assume that*

$$(P_{11}) \quad \min f_0 = \infty \quad \textit{and} \quad \max f_\infty = \beta_1 \in \left(0, \frac{1}{B\omega}\right).$$

Then (1.1) *has at least one positive* ω*-periodic solution.*

Proof. Let $\Omega_{\rho_0} = \{y \in X : \|y\| < \rho_0\}$. At first, in view of $\min f_0 = \infty$, we know from the proof of Theorem 2.2.4 that: $\|\Phi y\| \geqslant \|y\|$ for any $y \in E \cap \partial\Omega_{\rho_0}$.

Take $\Omega_{\rho_1} = \{y \in X : \|y\| < \rho_1\}$. Since $\max f_\infty = \beta_1 \in (0, 1/B\omega]$ from the proof of Theorem 2.7, we get

$$f(t,u) \leqslant \frac{\rho_1}{B\omega} \quad \text{for } |u| \leqslant \rho_1$$

and

$$\begin{aligned}(\Phi y)(t) &\leqslant B \int_t^{t+\omega} f\Bigg(s, \Bigg(y(g_1(s)), y(g_2(s)) \ldots y(g_{n-1}(s)), \\ &\qquad \int_{-\infty}^{s} k(s-\theta) y(\theta)\, \mathrm{d}\theta\Bigg)\Bigg)\, \mathrm{d}s \\ &\leqslant B \frac{\rho_1}{B\omega} \omega = \rho_1 = \|y\|,\end{aligned}$$

which implies $\|\Phi y\| \leqslant \|y\|$ for any $y \in E \cap \partial\Omega_{\rho_1}$. Hence, the claim is valid. □

By carrying out similar arguments as in Theorem 2.10, which bases on the discussion in subsection 2.2 and 2.3 one can easily reach the following three theorems.

Theorem 2.11. *Assume that*

$$(P_{12}) \quad \max f_\infty = 0 \quad \textit{and} \quad \min f_0 = \alpha_1 \in \left(\frac{1}{A\delta\omega}, \infty\right).$$

Then (1.1) *has at least one positive* ω*-periodic solution.*

Theorem 2.12. *Assume that*

$$(P_{13}) \quad \max f_0 = 0 \quad \textit{and} \quad \min f_\infty = \alpha_2 \in \left(\frac{1}{A\delta\omega}, \infty\right).$$

Then (1.1) *has at least one positive* ω*-periodic solution.*

Theorem 2.13. *Assume that*

$$(P_{14}) \min f_\infty = \infty \quad \textit{and} \quad \max f_0 = \beta_2 \in \left(0, \frac{1}{B\omega}\right).$$

Then (1.1) *has at least one positive* ω*-periodic solution.*

Theorem 2.14. *Assume that* (P_6) *holds. Moreover, if*

$$(P_{15}) \quad \min f_0 = \infty \quad \textit{and} \quad \min f_\infty = \alpha_3 \in \left(\frac{1}{A\delta\omega}, \infty\right).$$

Then (1.1) *has at least two positive* ω*-periodic solutions* y_1 *and* y_2 *satisfying* $0 < \|y_1\| \leqslant r_2 \leqslant \|y_2\|$, *where* r_2 *is defined in* (P_6).

Proof. Let $\Omega_{\rho_*} = \{y \in X : \|y\| < \rho_*\}$, where$\rho_* < r_2$. At first, in view of $\min f_0 = \infty$, we know from the proof of Theorem 2.1 that: $\|\Phi y\| \geqslant \|y\|$ for any $y \in E \cap \partial\Omega_{\rho_*}$.

Set $\Omega_{r_1} = \{y \in X : \|y\| < r_1\}$. Since $\min f_\infty = \alpha_3 \in (1/A\delta\omega, \infty)$, from the proof of Theorem 2.6, we get

$$f(t,u) \geqslant \frac{r_1}{A\omega} \quad \text{for } |u| \in [\delta r_1, r_1].$$

Incorporating (P_6) and the proof of Theorem 2.5, one can easily prove that (1.1) has at least two positive ω-periodic solutions y_1 and y_2 satisfying $0 < \|y_1\| \leqslant r_2 \leqslant \|y_2\|$. □

Theorem 2.15. *Assume* (P_6) *holds. Moreover, if*

$$(P_{16}) \quad \min f_\infty = \infty \quad \textit{and} \quad \min f_0 = \alpha_4 \in \left(\frac{1}{A\delta\omega}, \infty\right).$$

Then (1.1) *has at least two positive* ω*-periodic solutions* y_1 *and* y_2 *such that* $0 < \|y_1\| \leqslant r_2 \leqslant \|y_2\|$, *where* r_2 *is defined in* (P_6).

Theorem 2.16. *Assume that* (P_4) *holds. Moreover, if*

$$(P_{17}) \quad \max f_0 = 0 \quad \textit{and} \quad \max f_\infty = \beta_3 \in \left(0, \frac{1}{B\omega}\right).$$

Then (1.1) *has at least two positive* ω*-periodic solutions* y_1 *and* y_2 *such that* $0 < \|y_1\| \leqslant r_1 \leqslant \|y_2\|$, *where* r_1 *is defined in* (P_4).

Theorem 2.17. *Assume* (P_4) *holds. Moreover, if*

$$(P_{18}) \quad \max f_\infty = 0 \quad \textit{and} \quad \max f_0 = \beta_4 \in \left(0, \frac{1}{B\omega}\right).$$

Then (1.1) *has at least two positive* ω*-periodic solutions* y_1 *and* y_2 *such that* $0 < \|y_1\| \leqslant r_1 \leqslant \|y_2\|$, *where* r_1 *is defined in* (P_4).

Remark 2.1. Similar to Theorem 2.14, one can easily achieve Theorems 2.15, 2.16 and 2.17. The details of the proof are therefore omitted. In addition, from the proof of the above theorems, one can easily find that, in (P_7), (P_{10}), (P_{11}), (P_{14}), (P_{17}) and (P_{18}), $(0, 1/B\omega)$ can be replaced with $[0, 1/B\omega)$.

3. Existence of periodic solution of (1.2)

Now, we are at the position to attack the existence of positive periodic solution of (1.2). By carrying out similar arguments as in Section 2, it is not difficult to establish sufficient criteria for the existence of positive periodic solutions of (1.2). For simplicity, we prefer to list below the corresponding criteria for (1.2) without proof since the proofs are very similar those in Section 2.

Define

$$H(t,s) = \frac{\exp\{-\int_t^s a(\xi)\,d\xi\}}{1 - \exp\{-\int_0^\omega a(\xi)\,d\xi\}} = \frac{\exp\{\int_s^{t+\omega} a(\xi)\,d\xi\}}{\exp\{\int_0^\omega a(\xi)\,d\xi\} - 1}, \quad t, s \in R,$$

then from the definition it follows that

$$B = G(0,\omega) = H(t,t) \geqslant H(t,s) \geqslant H(t,t+\omega) = H(0,\omega) = G(t,t) = A$$

and

$$1 \geqslant \frac{H(t,s)}{H(t,t)} \geqslant \frac{H(t,t+\omega)}{H(t,t)} = \frac{A}{B} = \delta.$$

Lemma 3.1. *$y(t)$ is an ω periodic solution of Eq.* (1.2) *if and only if it is also an ω periodic solution of the following integral equation*:

$$y(t) = \int_t^{t+\omega} H(t,s) f\left(s, \left(y(g_1(s)), y(g_2(s)) \ldots y(g_{n-1}(s)), \int_{-\infty}^{s} k(s-\theta) y(\theta)\,\mathrm{d}\theta\right)\right) \mathrm{d}s. \tag{3.1}$$

Theorem 3.1. *Assume that one of the following conditions holds*:

(P_1); (P_2); (P_{11}); (P_{12}); (P_{13}); (P_{14}); (P_4) *and* (P_6); (P_7) *and* (P_8); (P_9) *and* (P_{10}).

Then (1.2) *has at least one ω-periodic positive solution.*

Theorem 3.2. *Assume that (P_4) holds. Moreover, if one of the following conditions holds*:

(P_3); (P_{17}); (P_{18}); (P_7) *and* (P_{10}).

Then (1.2) *has at least two positive ω-periodic solutions y_1 and y_2 such that* $0 < \|y_1\| < r_1 < \|y_2\|$, *where r_1 is defined in (P_4).*

Theorem 3.3. *Assume that (P_6) holds. Moreover, if one of the following conditions holds*:

(P_5); (P_{15}); (P_{16}); (P_8) *and* (P_9).

Then (1.2) *has at least two positive ω-periodic solutions y_1 and y_2 such that* $0 < \|y_1\| < r_2 < \|y_2\|$, *where r_2 is defined in (P_6).*

4. Examples and numerical simulations

In order to illustrate the generality and applicability of the main results, we present here some applications to several scalar differential equations models in biology, which have been extensively studied in the literature. In the following discussion, we will use the notations:

$$f^M = \max_{t\in[0,\omega]} f(t), \qquad f^m = \min_{t\in[0,\omega]} f(t),$$

where *f* is an ω-periodic function from $\mathbb{R}$ to $\mathbb{R}$. In addition, the parameters in the considered models are assume to be not equivalent to zero.

Example 4.1. Consider the generalized Logistic model of single species

$$\dot{x}(t) = x(t)\left[a(t) - \sum_{i=1}^{n} b_i(t)x(t-\tau_i(t)) - c(t)\int_{-\infty}^{t} k(t-s)x(s)\,\mathrm{d}s\right], \tag{4.1}$$

where $a, b_i, \tau_i \in C(\mathbb{R}, \mathbb{R}_+)$ are ω-periodic and $k : \mathbb{R}_+ \to \mathbb{R}_+$ is integrable such that $\int_0^{\omega} k(s)\,\mathrm{d}s = 1$.

Some special cases of (4.1) have been widely investigated in the literature, see, e.g., [4,8,9,11,12,19].

Theorem 4.1. (4.1) *has at least one positive* ω*-periodic solution.*

Proof. Note that

$$f(t,u) = -u_0\left(\sum_{i=1}^{n} b_i(t)u_i + c(t)u_{n+1}\right), \quad u = (u_0, \ldots, u_{n+1}) \in \mathbb{R}_+^{n+2}.$$

It is clear that (H_1)–(H_3) are satisfied. Moreover,

$$\max_{t\in[0,\omega]} \frac{|f(t,u)|}{|u|} \leqslant \left(\sum_{i=1}^{n} b_i^M + c^M\right)|u| \to 0 \quad \text{as } |u| \to 0,$$

whence $\max f_0 = 0$. In addition, if $u \in \mathbb{R}_+^{n+2}$ and $u_i > \delta|u|$, then

$$\min_{t\in[0,\omega]} \frac{|f(t,u)|}{|u|} \geqslant \delta^2\left(\sum_{i=1}^{n} b_i^m + c^m\right)|u| \to +\infty \quad \text{as } |u| \to +\infty,$$

which implies $\min f_\infty = \infty$. We have shown that (P_2) is satisfied. The claim follows from Theorem 3.1 directly. □

By carrying out similar arguments as above, one can easily show that any of the following generalized single species models admits at least one positive ω-periodic solution:

(i) The logistic equation of multiplicative type with several discrete delays

$$x'(t) = x(t)\left[a(t) - \prod_{i=1}^{n} b_i(t)x(t-\tau_i(t))\right],$$

where a, b_i, τ_i have the same assumptions as those in (4.1).

(ii) The generalized Richards single species growth model,

$$x'(t) = x(t)\left[a(t) - \left(\frac{x(t-\tau(t))}{K(t)}\right)^{\theta}\right], \tag{4.2}$$

where $a, K, \tau \in C(\mathbb{R}, \mathbb{R}_+)$ are ω-periodic and $\theta > 0$ is constant. If a, K are positive constants and $\tau \equiv 0$, then (4.2) is the original model proposed by Richards [17], which is also known as the Gilpin–Ayala model.

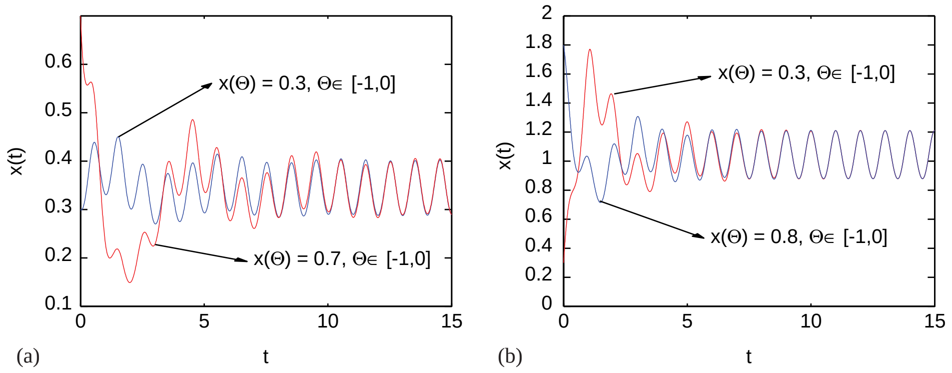

Fig. 1. Numerical simulations of solutions of (4.1) and (4.3): (a) $x'(t) = x(t)[1+\sin(2\pi t) - (3+\cos(2\pi t))x(t-1)]$; (b) $x'(t) = -(1+\sin(2\pi t))x(t) + (3+\cos(2\pi t))\exp\{-x(t-1)\}$.

(iii) The generalized so-called Michaelis–Menton type single species growth model

$$x'(t) = x(t)\left[a(t) - \sum_{i=1}^{n} \frac{b_i(t)x(t-\tau_i(t))}{1 + c_i(t)x(t-\tau_i(t))}\right],$$

where $a, b_i, c_i, \tau_i \in C(\mathbb{R}, \mathbb{R}_+)$ are ω-periodic.

Example 4.2. Consider the generalized hematopoiesis model

$$x'(t) = -\gamma(t)x(t) + \alpha(t)\exp\{-\beta(t)x(t-\tau(t))\}, \tag{4.3}$$

where $x(t)$ is the number of red blood cess at time t, and $\gamma, \alpha, \beta \in C(\mathbb{R}, \mathbb{R}_+)$ are ω-periodic. If $\gamma, \alpha, \beta, \tau$ are positive constants, then (4.3) is the original model introduced by Wazewska–Czyzewska and Lasota [21,22] modelling the survival of red blood cells in animal. The original version of (4.3) and its more general cases has been studies by some authors [2,6,13], see, for example, Chow [2] studied the periodicity of the original Wazewska–Lasota model.

Theorem 4.2. (4.3) *has at least one positive ω-periodic solution* (*Fig.* 1).

Proof. Obviously, (H_1)–(H_3) are satisfied, and

$$f(t,u) = \alpha(t)\exp\{-\beta(t)u\}, \quad u \in \mathbb{R}_+.$$

Then

$$\min_{t\in[0,\omega]} \frac{|f(t,u)|}{|u|} = \min_{t\in[0,\omega]} \frac{\alpha(t)}{u\exp\{\beta(t)u\}} \geqslant \frac{\alpha^m}{u\exp\{\beta^M u\}} \to +\infty \quad \text{as } u \to 0,$$

$$\max_{t\in[0,\omega]} \frac{|f(t,u)|}{|u|} \leqslant \frac{\alpha^M}{u\exp\{\beta^m u\}} \to 0 \quad \text{as } u \to +\infty,$$

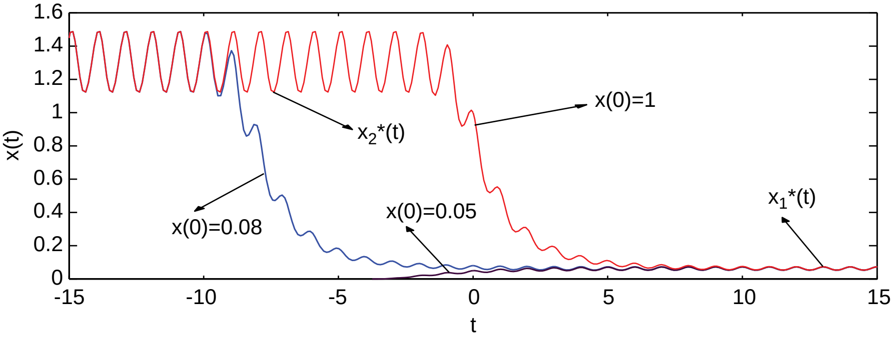

Fig. 2. Numerical simulations of solutions of (4.4) with $\alpha=\frac{1}{2}$ and $\beta=5$. The solution curves satisfying $x(0)=0.05$, $x(0)=0.08$ and $x(0)=1$ are illustrated in extended phase space, i.e., $x(t)$ plotted against t. It is clear that $x_1^*(t)$ is stable while $x_2^*(t)$ is unstable.

whence $\min f_0 = \infty$ and $\max f_\infty = 0$, i.e., (P_1) is satisfied. Theorem 2.1 proves the claim. □

Example 4.3. Consider the scalar ordinary differential equation

$$x'(t) = -(1+\sin(2\pi t))x(t) + (\tfrac{1}{4} + \tfrac{1}{32}\cos(2\pi t))(x^\alpha(t) + x^\beta(t)), \tag{4.4}$$

where $0<\alpha<1$ and $\beta>1$ are constant.

It is clear that

$$f(t,u) = \left(\tfrac{1}{4} + \tfrac{1}{32}\cos(2\pi t)\right)(u^\alpha + u^\beta), \quad u \geqslant 0$$

and

$$\min_{t\in[0,1]} \tfrac{|f(t,u)|}{|u|} = \tfrac{7}{32}(u^{\alpha-1} + u^{\beta-1}),$$

then, it follows that $\min f_0 = \min f_\infty = \infty$, i.e., (P_5) is valid for (4.4). Let $r_2 = 1$, then for any $0 \leqslant u \leqslant r_2$, we have

$$f(t,u) \leqslant \frac{9}{16} \approx 0.5625 \leqslant \frac{r_2}{B\omega} = 1 - \mathrm{e}^{-1} \approx 0.6321,$$

which proves (P_6). By Theorem 2.4, we conclude that (4.4) has at least two positive 1-periodic solution $x_1^*(t)$ and $x_2^*(t)$ such that $0 < \|x_1^*\| < 1 < \|x_2^*\|$.

Numerical simulations strongly support the analytical conclusion, i.e. (4.4), admits two positive 1-periodic solutions (see Fig. 2). In Fig. 2, we illustrate three solution curves with initial value 0.05, 0.08 and 1, respectively. It is clear that $x_1^*(t)$ is stable while $x_2^*(t)$ is unstable. The unstable 1-periodic solution $x_2^*(t)$ is numerically found by plotting the backward continuation of the solutions of (4.4) through (0, 0.08) and (0, 1).

Example 4.4. The coming example is also concerned with the regulation of hematopoiess, the formation of blood cell elements in the body. The model is originally due to Mackey and

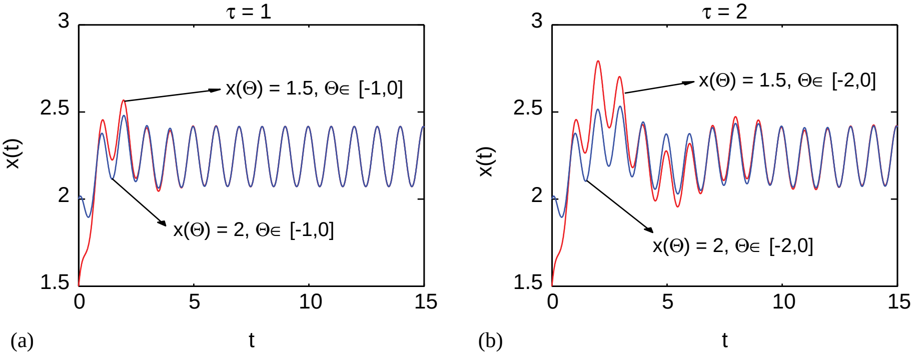

Fig. 3. Numerical simulations of solutions of (4.5) with $\gamma(t)=1+0.5\sin(2\pi t)$, $\beta(t)=6+0.2\cos(2\pi t)$, $\tau(t)\equiv\tau$, $\theta=1$, $n=2$. Numerical simulations show that the 1-periodic solution is stable and the solutions are more oscillatory as the time delay τ increases.

Glass [14]. Here we consider a generalization of the original one, where some parameters are generalized from positive constants to nonnegative ω-periodic functions, that is,

$$x'(t) = -\gamma(t)x(t) + \frac{\beta(t)\theta^n x(t-\tau(t))}{\theta^n + x^n(t-\tau(t))}, \tag{4.5}$$

where $\gamma, \beta, \tau \in C(\mathbb{R}, \mathbb{R}_+)$ are ω-periodic, θ is a positive constant and n is a given positive integer. In (4.5), $x(t)$ denotes the concentration of mature cells in the circulating blood and τ is the time delay between the production of immature cells in the bone marrow and their maturation for the release in the circulating bloodstream, please see [14] for the details of the derivation of the model.

Theorem 4.3. *If $\beta^m > (1-\delta)/\delta^2\omega$, then* (4.5) *has at least one positive ω-periodic solution, where $\delta = \exp\{-\int_0^\omega \gamma(s)\,\mathrm{d}s\}$.* (*see Fig.* 3)

Proof. Note that

$$f(t,u) = \frac{\beta(t)\theta^n u}{\theta^n + u^n}, \quad \max_{t\in[0,\omega]} \frac{|f(t,u)|}{|u|} = \frac{\beta^M\theta^n}{\theta^n+u^n},$$
$$\min_{t\in[0,\omega]} \frac{|f(t,u)|}{|u|} = \frac{\beta^m\theta^n}{\theta^n+u^n}, \quad u \geqslant 0.$$

If $\beta^m > (1-\delta)/\delta^2\omega$, then we have $\max f_\infty = 0$, $\min f_0 = \beta^m > 1/A\delta\omega$. By Theorem 2.11, we conclude that (4.5) has at least one positive ω-periodic solution. The proof is complete. □

In order to describe some physiological control systems, Mackey and Glass [14] also proposed another nonlinear delay differential equation (DDE) as their appropriate model. Now, we consider its nonautonomous version, where the parameters are periodic functions

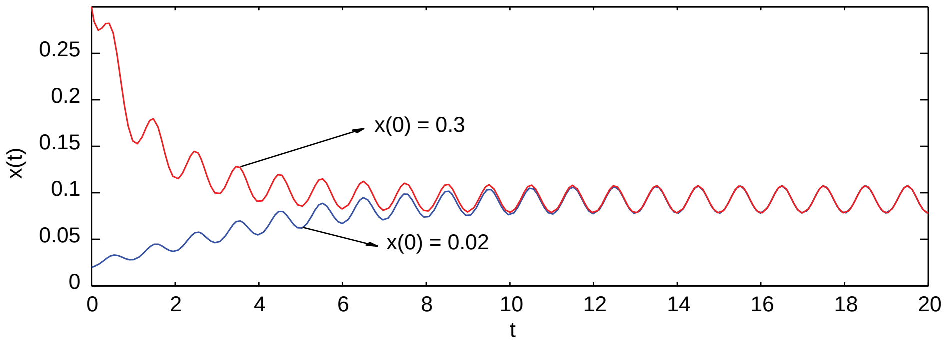

Fig. 4. Numerical simulations of solutions of (4.7) with $a(t)=1+\sin(2\pi t)$, $b(t)=6+\cos(2\pi t)$, $qE=0.5$, $c=1$.

of time t instead of positive constants in the original model:

$$x'(t) = -\gamma(t)x(t) + \frac{\beta(t)}{1+x^n(t-\tau(t))}, \tag{4.6}$$

where $\gamma, \beta, \tau \in C(\mathbb{R}, \mathbb{R}_+)$ are ω-periodic, and $n > 0$ are given integers. Similar to the above discussion, one can easily reach the following claims.

Theorem 4.4. (4.6) *has at least one positive ω-periodic solution* (by Theorem 2.1).

Example 4.5. Consider a single species population, whose growth law obeys the generalized Michaelis–Menton type growth equation

$$x'(t) = x(t)\left[a(t) - \frac{b(t)x(t)}{1+cx(t)}\right],$$

where $a(t), b(t) \in C(\mathbb{R}, \mathbb{R}_+)$ are ω-periodic, and c is a positive constant. Now we assume that the population subjects to harvesting. Under the catch-per-unit-effort hypothesis [3], the harvested population's growth equation reads

$$x'(t) = x(t)\left[a(t) - \frac{b(t)x(t)}{1+cx(t)}\right] - qEx(t), \tag{4.7}$$

where q and E are positive constants denoting the catch ability coefficient and the harvesting effort, respectively.

Theorem 4.5. *If*

$$0 < qE < \frac{1-\delta}{\omega}, \qquad \left(\frac{b^m}{c} + qE\right) > \frac{1-\delta}{\delta^2\omega},$$

then (4.7) *has at least one positive ω-periodic solution, where* $\delta = \exp\{-\int_0^\omega a(s)\,\mathrm{d}s\}$ (*see Fig.* 4).

Proof. In this case,

$$f(t,u)=\frac{b(t)u^2}{1+cu}+qEu, \quad u\geqslant 0.$$

It is trivial to show that

$$\max f_0=qE, \quad \min f_\infty=\frac{b^m}{c}+qE.$$

The conditions in Theorem 4.5 guarantee that (P_7) and (P_8) hold, then the conclusion directly follows from Theorem 3.1. □

Example 4.6. Consider the generalized so-called Nicholson's Blowflies Model

$$x'(t)=-\gamma(t)x(t)+\beta(t)x(t-\tau(t))\exp\{-ax(t-\tau(t))\}, \tag{4.8}$$

where γ, β, $\tau\in C(\mathbb{R},\mathbb{R}_+)$ are ω-periodic and a is a positive constant. When all the parameters are positive constants (4.8) reduces to the original model developed by Gurney et al. [7] to describe the population of the Australian sheep-blowfly (Lucila Curpina) that agrees well with the experimental data of Nicholson [15]. Since the equation explains Nicholson's data of blowfly quite accurately, it is now referred to as the 'Nicholson's Blowflies Model'.

The variation of the environment plays an important role in many biological and ecological dynamical systems. In particular, the effects of a periodically varying environment are important for evolutionary theories, as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Thus, the assumption of periodicity of the parameters of the system (in a way) incorporates the periodicity of the environment (e.g., seasonal effects of weather, food supplies, mating habits, etc.). In fact, it has been suggested by Nicholson [16] that any periodic change of climate tends to impose its periodicity upon oscillations of internal origin or to cause such oscillations to have a harmonic relation to periodic climatic changes. In view of this, it is realistic to assume that the parameters in the models are periodic functions of period ω.

Theorem 4.6. *If* $\beta^m>(1-\delta)/\delta^2\omega$, *then* (4.8) *has at least one positive* ω*-periodic solution, where* $\delta=\exp\{-\int_0^\omega\gamma(s)\,\mathrm{d}s\}$.

The proof is exactly the same as that of Theorem 4.3, the details are omitted here.

Remark 4.1. The numerical simulations for solutions of the test equations or models are based on the ODEs IVP solver `ode23` and the DDEs IVP solver `dde23` in MATLAB [18].

5. Conclusive remarks

In this paper, we have systematically explored the existence and multiplicity of positive periodic solutions of scalar functional differential equations (1.1) and (1.2) which are general enough to incorporate as special cases many famous mathematical models. The approach is based on the application of the famous Krasnoselskii fixed point theorem, which is very

Table 1
The number of positive periodic solutions of (1.1) and (1.2)

NPPS	min $f_\infty = \infty$	min $f_\infty = 0$	min $f_\infty \notin D$	max $f_\infty = \infty$	max $f_\infty = 0$	max $f_\infty \notin D$
min $f_0 = \infty$	2	?	2	?	1[a]	1
min $f_0 = 0$	?	?	?	?	?	?
min $f_0 \notin D$	2	?	2	?	1	1
max $f_0 = \infty$	?	?	?	?	?	?
max $f_0 = 0$	1[a]	?	1	?	2	2
max $f_0 \notin D$	1	?	1	?	2	2

$D = \{0, +\infty\}$, the set consists of 0 and $+\infty$.
NPPS stands for the number of positive periodic solutions.
? represents the cases that we do not explored, which remain open yet.
[a] Denotes the cases investigated by [8,9,19], i.e., superlinear and sublinear case.

powerful and effective in dealing with the existence problems. Some new, more general and better sufficient criteria are established, which improve and generalize some related results existing in the literatures (e.g., [1,4,8,9,12,19,20]). It should be pointed out that several authors have explored some special form of (1.1) and (1.2) based on the Krasnoselskii fixed point theorem (e.g., [1,8,9,19,20]). For example, Cheng and Zhang [1] and Wang [20] investigate the existence of positive periodic solution of the following two scalar delay differential equations:

$$y'(t) = -a(t)y(t) + \lambda h(t) f(y(t - \tau(t))),$$
$$y'(t) = a(t)y(t) - \lambda h(t) f(t - \tau(t))),$$

which are clearly special cases of (1.1) and (1.2). The discussions in [1,8,9,19,20] only concern with several cases of ours (see examples in Table 1). In addition, Wang [20] also establish two criteria for the nonexistence of periodic solutions.

Our discussion are carried out based on four key numbers, i.e., max f_0, min f_0, max f_∞, min f_∞, which can take 0, ∞ or finite values. It is clear that based on these four numbers, one has to deal with 36 cases (see Table 1). Our whole discussion covers 16 of 36 cases and is divided into three parts (see Section 2), while the rest still remain open, which are very interesting but more challenging. Maybe, one has to find some other more powerful and effective tools and methods.

Our numerical simulations strongly support the analytical achievements. Although our theoretical studies cover nothing about the stability of positive periodic solutions, the numerical simulations show that the positive periodic solution of the test equations or models is stable. It seems that our concise criteria guarantee not only the existence of positive periodic solution but also its stability.

Acknowledgements

This work is finished when the second author is visiting the department of mathematics and statistics at Arizona State University (i.e. ASU) in USA. He would like to express his gratitude to Prof. Yang Kuang in ASU for his hospitality, valuable discussions and excellent suggestions that greatly improved the presentation of this work.

References

[1] S.S. Cheng, G. Zhang, Existence of positive periodic solutions for non-autonomous functional differential equations, EJDE 59 (2001) 1–8.
[2] S.N. Chow, Existence of periodic solutions of autonomous functional differential equations, 15 (1974) 350–378.
[3] C.W. Clark, Mathematical Bioeconomics: The Optimal Management of Renewable Resources, second ed., Wiley, New York, 1990.
[4] M. Fan, K. Wang, Optimal harvesting policy for single population with periodic coefficients, Math. Biosci. 152 (2) (1998) 165–177.
[5] H.I. Freedmand, J.H. Wu, Periodic solutions of single-species models with periodic delay, SIAM J. Math. Anal. 23 (1992) 689–701.
[6] K. Gopalsamy, S.I. Trofimchuk, Almost periodic solutions of Lasota–Wazewska type delay differential equations, J. Math. Anal. Appl. 237 (1999) 106–127.
[7] W.S. Gurney, S.P. Blythe, R.M. Nisbet, Nicholson's blowflies revisited, Nature 287 (1980) 17–21.
[8] D.Q. Jiang, J.J. Wei, The existence of positive periodic solution of nonautonomous delay differential equation, China Ann. Math. 20A (6) (1999) 715–720.
[9] D.Q. Jiang, J.J. Wei, Existence of positive periodic solutions for Volterra intergo-differential equations, Acta Math. Sinica 21B (4) (2001) 553–560.
[10] M.A. Krasnoselskii, Positive Solution of Operator Equation[M], Gomingen, Noorhoff, 1964.
[11] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, Academic Press, Boston, 1993.
[12] Y.K. Li, Existence and global attractivity of positive periodic solutions for a class of delay differential equations, Chinese Sci. 28A (2) (1998) 108–118.
[13] J.W. Li, S.S. Cheng, Global attractivity in a survival model of Wazewska and Lasota, Quart. Appl. Math. 60 (3) (2002) 477–483.
[14] M.C. Mackey, L. Glass, Oscillation and chaos in physiological control system, Science 197 (1977) 287–289.
[15] A.J. Nicholson, The self adjustment of population to change, Cold Sprang Har'b. Syrup. Quant. Bzol. 22 (1957) 153–173.
[16] A.J. Nicholson, The balance of animal population, J. Animal Ecol. 2 (1993) 132–178.
[17] F.J. Richards, A flexible growth function for empirical use, J. Exp. Botany 10 (29) (1959) 290.
[18] L.F. Shampine, I. Gladwell, S. Thompson, Solving ODEs with Matlab, Cambridge University Press, Cambridge, 2003.
[19] A.Y. Wan, D.Q. Jiang, Existence of positive periodic solutions for functional differential equations, Kyush J. Math. 56 (2002) 193–202.
[20] H.Y. Wang, Positive periodic solutions of functional differential equations, J. Differential Equations 202 (2) (2004) 354–366.
[21] M. Wazewska-Czyzewska, A. Lasota, Mathematical models of the red cell system, Mat. Stosowana 6 (1976) 25–40.
[22] M. Wazewska-Czyzewska, A. Lasota, Mathematical problems of the dynamics of the red blood cells system, Annals of the Polish Mathematical Society Series III, Appl. Math. 17 (1976) 23–40.